The Density of States for Fermi Superfluids
with Layered Structures in the Order Parameter:
With Application to a Solution ^{3}He in ^{4}He at Low Temperatures

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The density of states of Fermionic excitations in *s*-wave Fermi superfluids is calculated. The order parameter is assumed to have a layered structure with a variation along only a single direction and the Fermions are allowed to move over the full three-dimensional space. The density of states for such a system is calculated using the Bogoliubov Hamiltonian in which the quadratic momentum dependence of the kinetic energy is approximated by a linear dependence. A representation of the density of states in terms of a Fredholm determinant is derived. For the case of a piecewise constant order parameter, the Fredholm determinant is calculated exactly. Explicit results for the density of states are presented for an order parameter with a periodic variation along a single direction. © 1993 Academic Press, Inc.

1. INTRODUCTION

Fermi superfluids such as superfluid ^{3}He, superconductors, the predicted (but so far unobserved) superfluidity of a dilute solution ^{3}He in ^{4}He are all characterised by an order parameter that plays the role of an off-diagonal potential. The Fermionic excitations move in the field of this potential and spatial variations of the order parameter have a significant effect upon the excitations. One way this is reflected is in the density of states which can be quite different to that of a spatially uniform system.

The present work has the aim of calculating the density of states for *s*-wave Fermi superfluids that are three-dimensional in extent but which contain a variation of the order parameter along only a single direction. Our interest lies in the layered structures in the order parameter that are expected to occur in ^{3}He–^{4}He mixtures in a magnetic field at ultralow temperatures [1], where the ^{3}He has become superfluid. Inhomogeneous order parameter structures were originally discussed by Larkin, Ovchinnikov, Fulde, and Ferrell [2] in the context of superconductivity in the presence of ferromagnetism. Spatially inhomogeneous order

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parameters result from the competition between the paramagnetic influence of a magnetic field and the tendency towards superfluid pairing.

Note that the density of states calculated here is not directly relevant to high T_c superconductors, where the electrons in such systems, while seeing a one-dimensional variation in the order parameter, additionally have their motion largely confined to planes (for recent work in this area see, e.g., Ref. [3]). By contrast we consider Fermions that are not confined to planes but are able to move over all of space. As a result the calculational methods used here are quite different to those of, e.g., Ref. [3].

The arrangement of this paper is as follows. In Section 2 we define the problem at hand. In Section 3 we show how the density of states is expressed in terms of a Fredholm determinant and this is evaluated, in Section 4, for piecewise constant order parameters. Sections 5, 6, and 7 are concerned with periodically varying order parameters and for a particular periodic order parameter the density of states is calculated. Section 8 consists of a short discussion and there are three appendices. The paper is written in units in which $\hbar = 1$.

2. DEFINITION OF THE PROBLEM

We consider an s-wave spin $\frac{1}{2}$ Fermi superfluid that extends spatially in three dimensions. It contains an order parameter with a variation only along a single direction, which is taken to be the x-axis.

The calculations that follow are made within the framework of a model mean field theory that describes the Fermionic excitations within the superfluid. In terms of the Pauli spin matrices ($\sigma_1, \sigma_2, \sigma_3$) that act in the space of spin up particles and spin down holes, the excitations are described by the first quantised Bogoliubov Hamiltonian:

$$H = K(\mathbf{p}) \, \sigma_3 + U(x). \tag{2.1a}$$

In this equation K is the kinetic energy measured relative to the Fermi energy $E_F = m v_F^2 / 2 \equiv k_F^2 / 2m$:

$$K(\mathbf{p}) = \mathbf{p}^2 / 2m - k_F^2 / 2m, \tag{2.1b}$$

and $U(x)$ is the of diagonal self-energy describing the superfluid. In terms of the s-wave order parameter $\Delta(x)$ and its complex conjugate $\Delta^*(x)$ we have

$$U(x) = \begin{pmatrix} 0 & \Delta(x) \\ \Delta^*(x) & 0 \end{pmatrix}; \tag{2.1c}$$

i.e., its matrix structure is a linear combination of σ_1 and σ_2.

We view the quantities $\mathbf{p}$ and x appearing in Eq. (2.1) as operators acting in the abstract spaces of Dirac bra and ket vectors; the operators being specified by their commutator algebra.

The density of states for the Fermionic excitations can be written, in a particle hole symmetry approximation as[1],

$$\rho(E) = \mathrm{Tr}_3[\delta(E - H)], \tag{2.2}$$

where Tr_3 denotes a combined trace over both matrix indices and the configuration space appropriate to a particle moving in three dimensions[2] and $\delta(E)$ denotes a Dirac delta function. The above equation for the density of states includes the contribution of both spin orientations of the Fermions. For a normal system, $U(x) = 0$ and in the particle hole symmetry approximation, the normal density of states is

$$\rho_N(E) = 2N(0)V \tag{2.3}$$

with

$$N(0) = mk_F/(2\pi^2) \tag{2.4}$$

and V is the volume of the system.

In order to calculate the density of states for the Hamiltonian H of Eq. (2.1) it is necessary to have essentially complete information on the spectrum of the operator in question. In general we do not know how to obtain this but in the next section we shall employ an approximation which allows us to make considerable progress.

3. EXPRESSING THE DENSITY OF STATES IN TERMS OF A FREDHOLM DETERMINANT

In Eq. (2.1) the quadratic momentum dependence of the kinetic energy makes the Hamiltonian a somewhat complicated operator [4]. When, however, the characteristic scale of spatial variation of the order parameter, ξ, obeys $1/(k_F\xi) \ll 1$ the quadratic momentum dependence of the kinetic energy may be well approximated by a linear momentum dependence. This is a simplifying approximation that we shall make in the present work. In Ref. [5] the essential details of the linearisation have been given; however, for completeness we present, in Appendix A, a derivation

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[1] Throughout this work we measure all energies, E, relative to the Fermi energy, E_F. The particle hole symmetry approximation entails working to zeroth order in E/E_F and we shall make this approximation throughout this work.

[2] More generally we shall write Tr_d for the combined trace over both matrix indices and configuration space appropriate to a particle moving in d spatial dimensions.

of the kinetic energy linearisation. The key result is Eq. (A.13) for the trace of a function f depending on the kinetic energy and x:

$$\mathrm{Tr}_3[f((\mathbf{p}^2 - k_{\mathrm{F}}^2)/2m; x)] \approx \frac{\pi N(0) A}{v_{\mathrm{F}}} \int_{-v_{\mathrm{F}}}^{v_{\mathrm{F}}} dc \, |c| \, \mathrm{Tr}_1[f(cp_x; x)], \qquad (3.1)$$

where A is area of the system in the y, z plane, Tr_1 is a combined trace over the matrix indices and configuration space of a particle moving along the x axis and p_x is the momentum operator for motion along this axis. The quantity c, with the dimension of velocity is, in fact, $v_{\mathrm{F}} \cos \vartheta$, where v_{F} is the Fermi velocity and ϑ may be thought of as the angle the momentum of an excitation makes with the x axis. Equation (3.1), when applied to the density of states leads to a particle hole symmetry approximation. For notational simplicity we write Tr for Tr_1, p for p_x, and obtain the following approximation to the density of states:

$$\rho(E) = \mathrm{Tr}_3[\delta(E - H)] = \frac{\pi N(0) A}{v_{\mathrm{F}}} \int_{-v_{\mathrm{F}}}^{v_{\mathrm{F}}} dc \, |c| \, \mathrm{Tr}[\delta(E - h)] \qquad (3.2a)$$

$$h = cp\sigma_3 + U(x). \qquad (3.2b)$$

Since $U(x)$ has a matrix structure, Eq. (3.2b) is identical to the Dirac Hamiltonian for a relativistic Fermion moving in one spatial dimension that possesses a spatially varying mass, the speed of light being given by c. A similar Hamiltonian is used to describe the linear molecule polyacetylene in the continuum limit [6]. The linearisation in the context of superfluidity therefore turns the calculation into one associated with a family (by virtue of the c integration) of one-dimensional polyacetylene-like problems.

When $U(x)$ is sufficiently simple we can directly use Eq. (3.2) to determine the density of states. In Appendix B we use Eq. (3.2) to calculate a result we shall refer to later; the density of states for the order parameter $\Delta(x) = \Delta \exp[-2ikx]$.

In more general cases, where h cannot be simply diagonalised, it may be helfpul to represent the density of states in terms of a Fredholm determinant which may then be calculated. To obtain this representation we introduce h_0, the Hamiltonian operator of a normal system obtained by setting $U(x) = 0$:

$$h_0 = cp\sigma_3. \qquad (3.3)$$

Using the density of states for this system from Eq. (2.3), we can write

$$\rho(E) = \frac{\pi N(0) A}{v_{\mathrm{F}}} \int_{-v_{\mathrm{F}}}^{v_{\mathrm{F}}} dc \, |c| \, \mathrm{Tr}[\delta(E - h) - \delta(E - h_0)] + 2N(0) V. \qquad (3.4)$$

It is the one-dimensional trace $\mathrm{Tr}[\delta(E - h) - \delta(E - h_0)]$ that can be represented in terms of a Fredholm determinant. To see this we first note that if E_n (E_{n0}) are the eigenvalues of h (h_0) then

$$\mathrm{Tr}[\delta(E - h) - \delta(E - h_0)] = \sum_n [\delta(E - E_n) - \delta(E - E_{n0})]. \qquad (3.5)$$

Next, with Im the imaginary part of a quantity, ln the principle branch of the logarithm,[3] and $\Theta(E)$ the Heaviside step function, we note that

$$\text{Im} \ln(E - E_n - i0_+) = -\pi\Theta(E_n - E). \tag{3.6}$$

It follows that we can represent the delta functions as

$$\delta(E - E_n) = \frac{1}{\pi} \frac{\partial}{\partial E} \text{Im} \ln(E - E_n - i0_+). \tag{3.7}$$

Lastly we note that Fredholm (or functional) determinant,[4]

$$D(\omega) = \text{Det}[(\omega - h)/(\omega - h_0)] \equiv \prod_n (\omega - E_n)/(\omega - E_{n0}), \tag{3.8}$$

contains, for $\omega = E - i0_+$, many factors of the form $(E - E_n - i0_+)$, one of which appears in the argument of the logarithm in Eq. (3.7). Consequently we have

$$\text{Tr}[\delta(E - h) - \delta(E - h_0)] = \frac{1}{\pi} \frac{\partial}{\partial E} \text{Im} \ln D(E - i0_+). \tag{3.9}$$

Using this result in Eq. (3.4) leads to the desired representation of the density of states in term of a Fredholm determinant:

$$\rho(E) = \frac{\partial}{\partial E} \left\{ \frac{N(0)A}{v_F} \int_{-v_F}^{v_F} dc \, |c| \, \text{Im} \ln D(E - i0_+) \right\} + 2N(0) V. \tag{3.10}$$

4. Method of Obtaining the Fredholm Determinant

The Fredholm determinant $D(\omega)$ given in Eq. (3.8) contains information on the full spectrum of the operator h of Eq. (3.2b) and as such it is a complicated object. In order to motivate the method by which we obtain $D(\omega)$ let us first consider the problem of determining the eigenvalues of the operator h. To make the problem well defined we impose a periodic boundary condition over an interval L. Thus on descending to coordinate representation, the eigenvalues of h are the values of E_n such that the two component spinor eigenfunction ψ obeys both

$$[-ic \, \partial_x \sigma_3 + U(x)] \, \psi(x) = E_n \psi(x) \tag{4.1a}$$

and

$$\psi(L) = \psi(0). \tag{4.1b}$$

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[3] The principle branch of the logarithm, ln, is defined on a complex plane cut along the negative real axis and has an imaginary part lying in $(-\pi, \pi)$.

[4] We use the symbol Det to denote a Fredholm determinant and det to denote the determinant of a finite matrix.

Since Eq. (4.1a) is a linear differential equation, we can use this equation to relate $\psi(L)$ to $\psi(0)$ and generally we find

$$\psi(x) = M(L; E_n)\, \psi(0), \tag{4.2}$$

where $M(L; E_n)$ is a 2×2 matrix.[5] The quantization condition Eq. (4.1b) can, using Eq. (4.2), be recast in the form

$$[1 - M(L; E_n)]\, \psi(0) = 0 \tag{4.3}$$

and the condition for a nonvanishing eigenfunction is

$$\det[1 - M(L; E_n)] = 0. \tag{4.4}$$

This is the condition that determines the eigenvalues E_n; the corresponding condition for the eigenvalues E_{n0} of h_0, obtained by setting $U(x) = 0$, is written as

$$\det[1 - M_0(L; E_{n0})] = 0. \tag{4.5}$$

It is interesting to consider, for general (complex) values of ω, the quantity $d(\omega)$ defined by

$$d(\omega) = \frac{\det[1 - M(L; \omega)]}{\det[1 - M_0(L; \omega)]}. \tag{4.6}$$

We hope to make it plausible, by the following arguments,[6] that $d(\omega)$ coincides precisely with the Fredholm determinant $D(\omega)$ given in Eq. (3.8). We note, for completeness, that the calculation of the Fredholm determinant for Schrödinger operators is well known in the physics literature [7].

Let us proceed now with the arguments for the identification of $d(\omega)$ and $D(\omega)$. We note:

(i)   Both the numerator and denominator of $d(\omega)$ are analytic functions of ω without any singularities in the finite complex ω plane. This follows since the matrices involved are of finite order and for finite L there is no route for infinities to enter into the problem.

(ii)   As a result of (i), zeros of $d(\omega)$ follow from the behaviour of the numerator and poles of $d(\omega)$ follow from the behaviour of the denominator. Let us assume that eigenvalues of h are all non-degenerate[7] and likewise h_0. Then by virtue

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[5] The eigenfunction $\psi(x)$ can, most generally, be written in terms of the path ordered exponential (with ordering operator P_x): $\psi(x) = P_x \exp(i \int_0^x dx'\, \sigma_3 [E_n - U(x')]/c)\, \psi(0)$ and hence we can identify $M(x; \omega) = P_x \exp(i \int_0^x dx'\, \sigma_3 [\omega - U(x')]/c)$.

[6] After completion of this work, a proof that $D(\omega) = d(\omega)$ has been found and will be presented elsewhere.

[7] In general some, or all, of the eigenvalues of h are degenerate (similarly h_0) but we can imagine adding small terms to h and h_0 whose only important effect is to render all eigenvalues of these operators non-degenerate.

of Eqs. (4.4) and (4.5) $d(\omega)$ has zeros only at the eigenvalues of h, E_n, and poles at the eigenvalues of h_0, E_{n0}. The property of both the numerator and denominator of Eq. (4.6) having zeroes at the respective eigenvalues allows us to write $d(\omega) = f(\omega) \prod_n (\omega - E_n)/(\omega - E_{n0}) \equiv f(\omega) D(\omega)$, where the assumed analytic function $f(\omega)$ has no poles or zeros in the finite complex plane[8] and $D(\omega)$ is the Fredholm determinant given in Eq. (3.8).

(iii) Lastly we note that when ω goes to infinity along any direction in the complex ω plane except the real axis, it may be explicitly verified that $d(\omega) \to 1$. In the same limit the Fredholm determinant $D(\omega) \to 1$ and we infer that in this limit $f(\omega) \to 1$. This is sufficient to show $f(\omega) = 1$ for all ω; the only analytic function without poles in the entire complex plane being a constant. Thus we identify the ratio of determinants of Eq. (4.6) with the Fredholm determinant of Eq. (3.8): $d(\omega) = D(\omega)$, i.e.,

$$D(\omega) \equiv \mathrm{Det}[(\omega - h)/(\omega - h_0)] = \frac{\det[1 - M(L; \omega)]}{\det[1 - M_0(L; \omega)]}. \tag{4.7}$$

Although the reasoning that led to this equation is not rigorous, we have tested it in non-trivial cases and believe that the result is generally correct. Granted this, a certain efficiency of calculation is achieved by its use; the ratio of two second-order determinants can be used to find the ratio of two infinite-order determinants.

5. Outline of the Calculation of the Density of States for Piecewise Constant Order Parameters

Equation (3.10) gives the density of states in terms of the Fredholm determinant $D(E - i0_+)$. By virtue of Eq. (4.7) we can determine the Fredholm determinant merely from knowledge of some 2×2 matrices and the evaluation of their determinant. In the case of a piecewise constant order parameter all the 2×2 matrices required for the calculation of $D(\omega)$ may be given in closed form and this is the class of order parameters that we shall consider in this work.[9]

[8] If the function $f(\omega)$ did have zeros or poles in the finite complex ω plane, these would either change the order of the zeros (poles) lying at the eigenvalues E_n (E_{n0}) or generate new zeros (poles).

[9] There is a possible objection in using a piecewise constant order parameter in Eq. (3.10) for the density of states. This comes about since the kinetic energy linearisation described in Appendix A (and used in the density of states) neglected some derivatives of the order parameter which are not obviously small for sharply varying order parameter structures. It turns out that the linearisation is perhaps better than expected. Calculations performed for a *single* discontinuous jump in the order parameter [8] indicate that the corrections to the eigenfunctions calculated with the linearised kinetic enegy are of order Δ/E_F, where Δ is a typical magnitude of the off-diagonal potential $U(x)$ appearing in Eq. (4.1a). It appears that the full second derivative character of the kinetic energy results in a small admixture of solutions that are omitted in the linearised theory. While these additional solutions may be important in questions of scattering they do not seem to be important as far as the free energy is concerned.

The calculation of $D(\omega)$, as follows from Eq. (4.7), requires the determination of the matrix $M(L; \omega)$. From Eq. (4.1a) it follows that the 2×2 matrix $M(x; \omega)$ obeys

$$[-ic \, \partial_x \sigma_3 + U(x)] \, M(x; \omega) = \omega M(x; \omega), \qquad M(0; \omega) = 1. \qquad (5.1)$$

Let us suppose that

$$
\begin{aligned}
U(x) = U_0 \qquad & x_1 > x > 0 \\
= U_1 \qquad & x_2 > x > x_1 \\
= U_2 \qquad & x_3 > x > x_2 \\
\vdots \qquad &
\end{aligned}
$$

and generally

$$U(x) = U_j \qquad x_{j+1} > x > x_j. \qquad (5.2)$$

Then rewriting Eq. (5.1) in the form

$$\partial_x M(x; \omega) = \frac{i}{c} [\sigma_3 \omega - \sigma_3 U(x)] \, M(x; \omega) \qquad (5.3)$$

and defining

$$N_j = \frac{i}{c} [\sigma_3 \omega - \sigma_3 U_j], \qquad (5.4)$$

we see that

$$M(x; \omega) = \exp[(x - 0) \, N_0], \qquad\qquad\qquad x_1 > x > 0$$

$$M(x; \omega) = \exp[(x - x_1) \, N_1] \exp[(x_1 - 0) \, N_0], \qquad\qquad x_2 > x > x_1$$

$$M(x; \omega) = \exp[(x - x_2) \, N_2] \exp[(x_2 - x_1) \, N_1] \exp[(x_1 - 0) \, N_0], \qquad x_3 > x > x_2,$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.5)$$

and in this way we can construct $M(x; \omega)$. This leads to an expression for $M(L; \omega)$ as a product of matrix exponentials and the density of states can then be found via Eqs. (4.7) and (3.10).

To understand the correct order of limiting procedures implicit in the above procedure, we use it to calculate the density of states for the simplest piecewise constant order parameter: that of a uniform system. We write this quantity as $\rho_0(E)$. The off-diagonal self-energy is

$$U(x) = \Delta \sigma_1. \qquad (5.6)$$

A short calculation results in[10]

$$D(\omega) = \frac{2 - 2\cosh((L/c)\sqrt{\Delta^2 - \omega^2})}{2 - 2\cosh((L/c)\sqrt{-\omega^2})} \qquad (5.7)$$

and for large L we have[11]

$$\ln D(\omega) \approx \frac{L}{|c|}\sqrt{\Delta^2 - \omega^2} - \frac{L}{|c|}\sqrt{-\omega^2}, \qquad (5.8)$$

leading to

$$\text{Im} \ln D(E - i0_+) = \{\text{sgn}(E)\sqrt{E^2 - \Delta^2}\,\Theta(E^2 - \Delta^2) - E\}\,L/|c|. \qquad (5.9)$$

Substituting this result in Eq. (3.10) yields the standard result

$$\rho_0(E) = 2N(0)\,V\frac{|E|}{\sqrt{E^2 - \Delta^2}}\,\Theta(E^2 - \Delta^2). \qquad (5.10)$$

6. Calculation of the Density of States for a Periodic Order Parameter

Having outlined the procedure for finding $M(L; \omega)$ and hence $D(\omega)$ for a general piecewise constant order parameter, we proceed now to look at the density of states for a *periodic* order parameter.

We assume that the potential repeats itself cyclically after a distance $2a$ and that there are n complete cycles in the length L of the system. Thus

$$L = 2na \qquad (6.1)$$

and we infer from Eq. (5.5) that

$$M(L; \omega) = [M(2a; \omega)]^n. \qquad (6.2)$$

The Fredholm determinant takes the form

$$D(\omega) = \frac{\det(1 - [M(2a; \omega)]^n)}{2 - 2\cosh((L/c)\sqrt{-\omega^2})}. \qquad (6.3)$$

We have noted in Section 2 that $U(x)$ is a linear combination of σ_1 and σ_2 and it

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[10] The results has been written in terms of $\sqrt{-\omega^2}$ and $\sqrt{\Delta^2 - \omega^2}$, where the square roots are cut along the negative real axis and the branch selected has a positive real part. This form anticipates the limit $L \to \infty$, where a branch cut results from states having a continuous spectrum. For finite L both the numerator and denominator are analytic functions of ω.

[11] We perform all calculations to leading order in the length of the system, L, ignoring any boundary effects and are, effectively working in the infinite volume limit. It is important to note that any large L approximations are to be made *before* the imaginary part of ω is taken to be infinitesimal.

follows that $M(2a; \omega)$ has a determinant of unity. Consequently the two eigenvalues of $M(2a; \omega)$ are reciprocals of each other and with hindsight we write

$$\text{Eigenvalues}[M(2a; \omega)] = \Lambda^2; \Lambda^{-2} \tag{6.4}$$

and

$$D(\omega) = \frac{2 - \Lambda^{2n} - \Lambda^{-2n}}{2 - 2\cosh((L/c)\sqrt{-\omega^2})}. \tag{6.5}$$

We denote by $\bar{\Lambda}^2$ the eigenvalue of $M(2a; \omega)$ which has the largest modulus (which is greater than unity). It follows that for large L,

$$\ln D(\omega) \approx \frac{L}{a} \ln \bar{\Lambda}(\omega) - \frac{L}{|c|} \sqrt{-\omega^2}. \tag{6.6}$$

On substituting this into Eq. (3.10) for the density of states, we find that the second term in Eq. (6.6) cancels with the normal state quantity $2N(0)V$ in Eq. (3.10) and we obtain, with $V = AL$, the volume of the system,

$$\rho(E) = \frac{\partial}{\partial E}\left\{ \frac{N(0)V}{v_F a} \int_{-v_F}^{v_F} dc \, |c| \, \text{Im} \ln \bar{\Lambda}(E - i0_+) \right\}. \tag{6.7}$$

Thus for a periodic system, the density of states is determined by the eigenvalue of maximum modulus of the 2×2 "transfer matrix" $M(2a; \omega)$ that propagates eigenfunctions across a single period of the potential. Since we have not yet made any assumptions about the continuity of the potential, this result applies not just for piecewise constant potentials but also for continuous ones.

Let us now focus on a particular periodic potential. We choose a real periodic order parameter with a "square wave" profile, $\Delta(x)$, which is given in Fig. 1:

$$\left. \begin{aligned} \Delta(x) &= -\Delta, & (2j+1)a > x > 2ja, \\ &= \Delta, & 2ja > x > (2j+1)a, \end{aligned} \right\} j = 0, 1, 2, ... \tag{6.8}$$
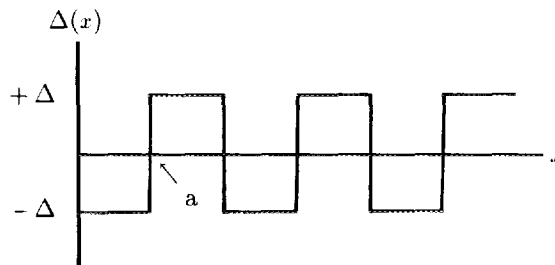

FIG. 1.   The profile of the "square wave" periodic order parameter $\Delta(x)$ given in Eq. (6.18) is plotted as a function of position. The period of oscillation is $2a$.

The matrix $M(2a; \omega)$ associated with a single period of this potential can be found in closed form. A straightforward calculation leads to

$$M(2a; \omega) = \frac{(\Delta^2 - \omega^2 \, \text{ch}(2t)) - \omega\Delta(\text{ch}(2t) - 1)\, \sigma_1 + i\omega \sqrt{\Delta^2 - \omega^2} \, \text{sh}(2t)\, \sigma_3}{\Delta^2 - \omega^2}, \qquad (6.9a)$$

where

$$t = \frac{a \sqrt{\Delta^2 - \omega^2}}{|c|}. \qquad (6.9b)$$

The eigenvalues of $M(2a; \omega)$ can be written as $\Lambda^{\pm 2}$, where

$$\Lambda^{\pm 1} = \frac{\sqrt{\Delta^2 - \omega^2 \, \text{ch}^2(t)} \pm i\omega \, \text{sh}(t)}{\sqrt{\Delta^2 - \omega^2}}. \qquad (6.10)$$

On replacing ω by $E - i0_+$, as is required in Eq. (6.7), the difference in the modulus of the eigenvalues is infinitesimal. We can save some work on finding which of the two eigenvalues has the largest modulus by using either of the eigenvalues; there is only a sign difference between $\partial \, \text{Im} \ln \Lambda(E - i0_+)/\partial E$ and $\partial \, \text{Im} \ln \Lambda^{-1}(E - iO_+)/\partial E$. Thus, since the density of states is positive, we can write Eq. (6.7) in an equivalent but more useful form by making the replacement $\partial \, \text{Im} \ln \bar{\Lambda}(E - i0_+)/\partial E$ by the modulus of $\partial \, \text{Im} \ln \Lambda(E)/\partial E$ (the infinitesimal $i0_+$ now playing no role). This leads to

$$\rho(E) = \left\{ \frac{N(0)V}{v_F a} \int_{-v_F}^{v_F} dc \, |c| \left| \frac{\partial}{\partial E} \, \text{Im} \ln \Lambda(E) \right| \right\}. \qquad (6.11)$$

In the two cases $E^2 < \Delta^2$, $E^2 > \Delta^2$ the behaviour of $\Lambda(E)$ is different and it is necessary to treat these separately. Before we do so, it is convenient to introduce the notation

$$\lambda = \sqrt{|E^2 - \Delta^2|} \qquad (6.12a)$$

$$r = a/|c| \qquad (6.12b)$$

and, since the density of states is symmetric in E, we restrict discussion to positive E.

Case (a). $E^2 < \Delta^2$. For this case, $\lambda = \sqrt{\Delta^2 - E^2}$ and we have

$$\Lambda(E) = \frac{\sqrt{\lambda^2 \, \text{ch}^2(r\lambda) - \Delta^2 \, \text{sh}^2(r\lambda)} + i \sqrt{\Delta^2 - \lambda^2} \, \text{sh}(r\lambda)}{\lambda}. \qquad (6.13)$$

From Eq. (6.11) it follows that for $\Lambda(E)$ to make a contribution to the density of states it must have a non-constant phase and from Eq. (6.13) we see that this

comes about by having $\sqrt{\lambda^2 \,\mathrm{ch}^2(r\lambda) - \Delta^2 \,\mathrm{sh}^2(r\lambda)}$ real, since $i\sqrt{\Delta^2 - \lambda^2}\,\mathrm{sh}(r\lambda)$ is always imaginary in this range of energies. The required reality of the square root can be written as the condition $\lambda^2 - \Delta^2 \,\mathrm{th}^2(r\lambda) > 0$ and we find[12]

$$\left| \frac{\partial}{\partial E} \,\mathrm{Im} \ln \Lambda(E) \right| = \Theta(\lambda^2 - \Delta^2 \,\mathrm{th}^2(r\lambda)) \frac{\partial}{\partial E} \arctan \left(\frac{\sqrt{\Delta^2 - \lambda^2}\,\mathrm{th}(r\lambda)}{\sqrt{\lambda^2 - \Delta^2 \,\mathrm{th}^2(r\lambda)}} \right). \quad (6.14)$$

It may be verified that the right-hand side of this equation is always $\geqslant 0$.

Case (b). $E^2 > \Delta^2$. For this case, $\lambda = \sqrt{E^2 - \Delta^2}$ and we have

$$\Lambda(E) = \frac{\sqrt{\lambda^2 \cos^2(r\lambda) - \Delta^2 \sin^2(r\lambda)} + i\sqrt{\Delta^2 + \lambda^2}\,\sin(r\lambda)}{\lambda} \quad (6.15)$$

and similar considerations to those of Case (a) lead to

$$\left| \frac{\partial}{\partial E} \,\mathrm{Im} \ln \Lambda(E) \right| = \Theta(\lambda^2 - \Delta^2 \tan^2(r\lambda)) \frac{\partial}{\partial E} \arctan \left(\frac{\sqrt{\Delta^2 + \lambda^2}\,\tan(r\lambda)}{\sqrt{\lambda^2 - \Delta^2 \tan^2(r\lambda)}} \right) \quad (6.16)$$

and, as in Eq. (6.14), the right-hand side is always $\geqslant 0$.

The density of states may now be obtained. We perform the differentiations indicated in Eqs. (6.14) and (6.16), substitute the results in Eq. (6.11), and go from the integration variable c (restricted to positive values) to the variable u given by

$$u = r\lambda \equiv a\lambda/c. \quad (6.17)$$

We obtain

$$\frac{\rho(E)}{2N(0)V} = \Theta(\Delta^2 - E^2) \frac{a\lambda^2}{v_\mathrm{F}} \int_{a\lambda/v_\mathrm{F}}^{\infty} \frac{du}{u^2} \frac{\Theta(\lambda^2 - \Delta^2 \,\mathrm{th}^2(u))}{\sqrt{\lambda^2 - \Delta^2 \,\mathrm{th}^2(u)}} \left[1 - \frac{\Delta^2}{\lambda^2}\left(1 - \frac{\mathrm{th}(u)}{u} \right) \right]$$

$$+ \Theta(E^2 - \Delta^2) \frac{a\lambda^2}{v_\mathrm{F}} \int_{a\lambda/v_\mathrm{F}}^{\infty} \frac{du}{u^2} \frac{\Theta(\lambda^2 - \Delta^2 \tan^2(u))}{\sqrt{\lambda^2 - \Delta^2 \tan^2(u)}} \left[1 + \frac{\Delta^2}{\lambda^2}\left(1 - \frac{\tan(u)}{u} \right) \right]$$

$$(6.18)$$

with λ given by Eq. (6.12a).

[12] We have implicity used continuity of $\mathrm{Im} \ln \Lambda(E)$ so there are no contributions from differentiating the Heaviside step function. $\mathrm{Im} \ln \Lambda(E)$ is, in fact, given by the continuous function

$$\mathrm{Im} \ln \Lambda(E) = \Theta(\lambda^2 - \Delta^2 \,\mathrm{th}^2(r\lambda)) \arctan \left(\frac{\sqrt{\Delta^2 - \lambda^2}\,\mathrm{th}(r\lambda)}{\sqrt{\lambda^2 - \Delta^2 \,\mathrm{th}^2(r\lambda)}} \right)$$

$$+ \Theta(\Delta^2 \,\mathrm{th}^2(r\lambda) - \lambda^2)\,\pi/2.$$

7. Behaviour of the Density of States for the Periodic Order Parameter

Equation (6.18) indicates that the density of states for for the "square wave" periodic order parameter has a rich structure. Before we delve into some of the details of this let us check that the result reduces to known results in various limiting cases:

(i)   For $\Delta \to 0$ we quickly find $\rho(E) \to \rho_N(E) = 2N(0)V$, the normal state result.

(ii)   For $E \to \infty$, i.e., $\lambda^2 \to \infty$, the size of the regions where $\Delta^2 \tan^2(u)$ is comparable or larger than λ^2 is very small. Neglecting these regions and replacing $\sqrt{\lambda^2 - \Delta^2 \tan^2(u)})$ by λ along with discarding terms of order λ^{-2}, leads again to the normal state result.

(iii)   A far less trivial limit is the behaviour of the density of states as $a \to \infty$. As the sign reversals of the order parameter become further apart, the Fermions, for most of the time, see only a constant order parameter. We therefore expect that in the $a \to \infty$ limit the density of states should go over to the result of a uniform system. In Appendix C we show how the density of states of a uniform system is approached as $a \to \infty$.

(iv)   It may be verified that in the limit $a \to 0$, the density of states of Eq. (6.18) goes over to the normal state result; the Fermions apparently seeing only an average of the order parameter in this limit (cf. the $k \to \infty$ in Appendix B).

Of the four limits discussed above, it seems that the third is by far the most severe, the density of states of Eq. (6.18) having no obvious resemblance to the density of states of a uniform system.

Let us now consider some other features of the density of states. We work with the dimensionless quantities

$$\bar{a} = a \, \Delta / v_{\mathrm{F}} \tag{7.1}$$

$$\varepsilon = E/\Delta \tag{7.2}$$

$$\bar{\rho}(\varepsilon) = \frac{\rho(E)}{2N(0)V}. \tag{7.3}$$

In Figs. 2a, b, c we have plotted the density of states of Eq. (6.18) as functions of energy for the parameter $\bar{a}$ taking the values 0.5, 1.0, 2.0. For aid of comparison we have plotted, in Fig. 3, the density of states for the order parameter $\Delta \exp[-2ikx]$, given in Eq. (B.5). The value of k selected; $k = \pi \, \Delta / 2v_{\mathrm{F}}$, corresponds to the same period of oscillation as that of the square wave order parameter appearing in Fig. 2b having $\bar{a} = 1$.
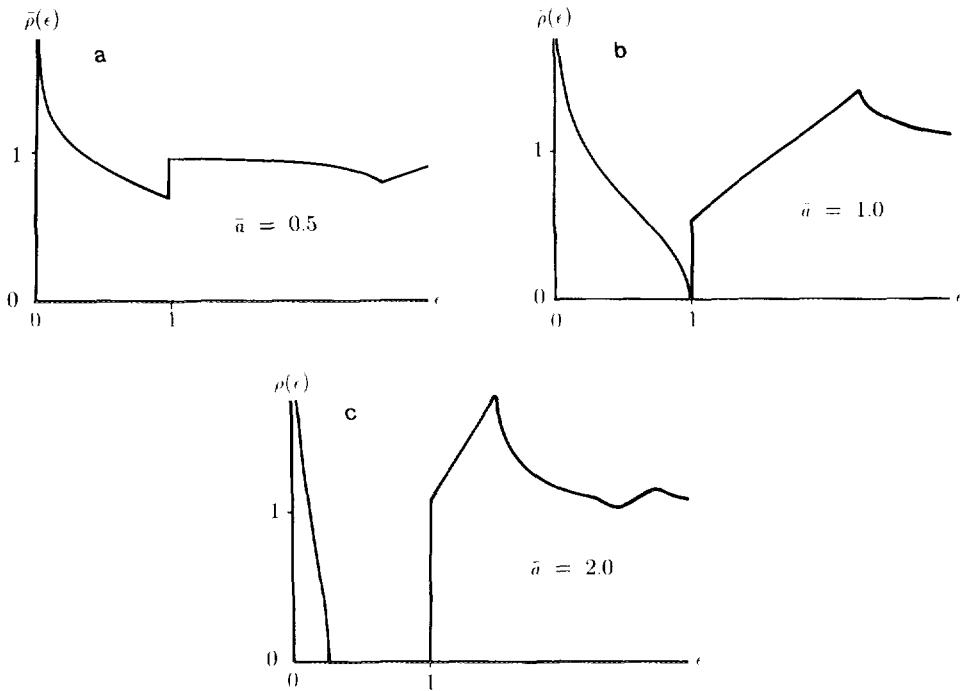
FIG. 2. The density of states is plotted as a function of energy for the "square wave" periodic order parameter, with profile $\Delta(x)$ given in Eq. (6.18) and plotted in Fig. 1. All quantities plotted have been rescaled to be dimensionless. $\bar{\rho}(\varepsilon)$ is the density of states measured in units of $2N(0)V$, ε is the energy measured in units of Δ, and $\bar{a}$ is the half period of the oscillation measured in units of v_F/Δ.
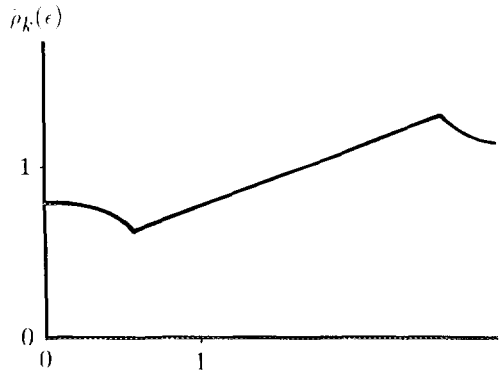
FIG. 3. The density of states is plotted as a function of energy for the complex order parameter $\Delta \exp(-2ikx)$. $\bar{\rho}_k(\varepsilon)$ is the density of states measured in units of $2N(0)V$ and ε is the energy measured in units of Δ. We have selected $k = \pi \Delta/2v_F$; this value of k makes the complex order parameter have the same period of oscillation as the real periodic order paramter of Fig. 2b.

It is clear from Figs. 2a, b, c that the density of states in the presence of the periodic order parameter is non-zero for some ε below $\varepsilon^2 = 1$, i.e., for energies below the magnitude of the order parameter $\varDelta$. This feature contrasts sharply with the density of states of a uniform system which is zero at energies below the gap.

The non-zero density of states of the periodic system "below the gap" comes about only because the order parameter passes through zero. If there were only a single sign change of the order parameter then there would exist a single bound state (for a given value of c) in direct analogy with the bound state on a sharp kink in polyacetylene [9]. The fact that there are many sign changes of the order parameter results in a band of states lying below the magnitude of the gap. It is possible to extract the analytical form of the density of states for $E^2 \ll \varDelta^2$; it is quite singular but does have a finite integral (as it must). It takes the form

$$\bar{\rho}(\varepsilon) = \frac{\pi \bar{a}}{2 |\varepsilon| \, [\ln(2/|\varepsilon|)]^3} \, (1 + O([\ln(2/|\varepsilon|)]^{-1})), \qquad \varepsilon \ll 1, \qquad (7.4)$$

the logarithmic corrections indicating that the leading term is a good approximation only for extremely small energies.

For the states lying above the magnitude of the gap, i.e., scattering states, the complexity of the expression for the density of states originates from the band structure associated with the Fermions moving in a periodic order parameter. Thus in Eq. (6.16) the Heaviside step function repeatedly switches on and off as E, and hence λ, increases. This indicates that there are, for a given value of c, band gaps in the spectrum. These are smeared out when we integrate over c, the integration summing the contributions of excitations whose momenta make different angles with the x axis.

Given that both real and complex periodic order parameters have been considered in investigations of inhomogeneous Fermi superfluids [1, 10], it is interesting to compare the density of states given in Fig. 2b for the real periodic order parameter and that in Fig. 3 for the complex order parameter $\varDelta \exp(-2ikx)$. The periods of oscillation of the two order parameters have been arranged to be the same by the value of k selected. There are significant differences in the way that spectral weight is distributed in the two cases and this is most obvious below the gap, indicating that although both are periodic they are far from equivalent. We believe this difference would persist even if the square wave order parameter were smoothed out.

A last feature we shall comment on is the discontinuity of the density of states as E^2 passes through $\varDelta^2$, i.e., as ε^2 passes through 1. For a uniform system, with constant order parameter, $\varDelta$, there is a square root singularity at $\varepsilon^2 = 1$ resulting in a divergence and infinite discontinuity of the density of states. The density of states for the periodic order parameter, is, by contrast finite at $\varepsilon^2 = 1$, there being only a finite discontinuity. In Figs. 2a, b, c this is discernible for all values of the

parameter $\bar{a}$. An analysis of Eq. (6.18) for the density of states leads, in general, to the result

$$\bar{\rho}(\varepsilon = 1_-) = \Theta(1 - \bar{a})\left(\sqrt{1 - \bar{a}^2} - \frac{\bar{a}}{3} \arccos \bar{a}\right), \tag{7.5}$$

$$\bar{\rho}(\varepsilon = 1_+) = \frac{\pi}{6}\bar{a} + \Theta(1 - \bar{a})\left(\sqrt{1 - \bar{a}^2} - \frac{a}{2} \arccos \bar{a}\right), \tag{7.6}$$

indicating that for all non-zero $\bar{a}$ there is a finite discontinuity and the divergence of the density of states of a uniform system at $\varepsilon = 1_+$ can only be reached in the $\bar{a} = \infty$ limit.

8. Discussion

In this work we have presented a method of calculating the density of states for superfluid Fermi systems possessing order parameters that very only along a single direction and are piecewise constant. The method centres about the calculation of a Fredholm determinant and we have given arguments for the identification of this quantity with the ratio of two finite order determinants.

We envisage a number of applications of the results in this paper; in particular we note that that in the absence of magnetic fields a knowledge of the Fredholm determinant is alone sufficient to determine the free energy [9, 11].

Our motivation of this work lies in using the density of states to calculate the free energy of a ^{3}He–^{4}He solution at low temperatures in the presence of a magnetic field and estimating the order parameter profile. We plan to present our findings on this elsewhere.

APPENDIX A: Linearisation of the Kinetic Energy

In this appendix we give the steps leading to a linearisation of the momentum dependence of the kinetic energy, an approximation used in Section 3. A related derivation has appeared in Ref. [5].

The linearisation is achieved by employing twice an exact relation for a function $g(\mathbf{p}; x)$ with a square matrix structure that depends on the both the x coordinate operator and the momentum operator $\mathbf{p}$, the latter being appropriate to a particle moving in d spatial dimensions. The relation is

$$\mathrm{Tr}_d\left[g(\mathbf{p}; x)\right] = \mathrm{tr} \int \frac{d^d k\, d^d x}{(2\pi)^d}\, g(\mathbf{k} - i\nabla; x), \tag{A.1}$$

where Tr_d denotes a combined trace over both the matrix indices and the configuration space appropriate to a particle moving in d spatial dimensions and tr denotes the trace over only the matrix degrees of freedom. The differential operator ∇ is taken to act to the right.

The proof of Eq. (A.1) is straightforward. We have

$$\langle \mathbf{x}'| \, g(\mathbf{p}; x) \, |\mathbf{x}''\rangle = g(-i\nabla'; x') \int d^dk/(2\pi)^d \exp[i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}'')]$$

$$= \int d^dk/(2\pi)^d \exp[i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}'')] \, g(\mathbf{k} - i\nabla'; x'). \qquad \text{(A.2)}$$

Equation (A.1) then follows by setting $\mathbf{x}'' = \mathbf{x}' = \mathbf{x}$, integrating each side over $\mathbf{x}$ and tracing over the matrix indices.

We now employ Eq. (A.1) in dimension $d = 3$ for a function $f((\mathbf{p}^2 - k_F^2)/2m; x)$ that depends on the kinetic energy and the x coordinate. We have

$$\mathrm{Tr}_3[f((\mathbf{p}^2 - k_F^2)/2m; x)]$$

$$= \mathrm{tr} \int \frac{d^3k \, d^3x}{(2\pi)^3} f((\mathbf{k}^2 - k_F^2)/2m - i\mathbf{k} \cdot \nabla/m - \nabla^2/2m; x). \qquad \text{(A.3)}$$

We set

$$\varepsilon = (\mathbf{k}^2 - k_F^2)/2m \qquad \text{(A.4)}$$

and make the approximations

$$\int d^3k/(2\pi)^3 \approx \int N(0) \, d\varepsilon \int d\Omega/4\pi \qquad \text{with} \quad N(0) = mk_F/2\pi^2 \qquad \text{(A.5)}$$

$$-i\mathbf{k} \cdot \nabla/m - \nabla^2/2m \approx -iv_F\hat{k} \cdot \nabla, \qquad \text{(A.6)}$$

where $d\Omega$ is the element of solid angle about the unit vector $\hat{k}$ which points in the direction of $\mathbf{k}$. Equation (A.5) simply neglects the variation of the free particle density of states at the Fermi energy while Eq. (A.6) neglects the variation of $\mathbf{k}$ in the vicinity of the Fermi energy and, furthermore, assumes the spatial variation of the order parameter is slow on a scale k_F^{-1}. We obtain

$$\mathrm{Tr}_3[f((\mathbf{p}^2 - k_F^2)/2m; x)] \approx \mathrm{tr} \int d^3x \int N(0) \, d\varepsilon \, f(\varepsilon - iv_F\hat{k} \cdot \nabla; x). \qquad \text{(A.7)}$$

The polar axis of $\hat{k}$ is orientated along the x axis such that

$$\hat{k} \cdot \hat{x} = \cos \vartheta \qquad \text{(A.8)}$$

and, since the only coordinate appearing in f is x,

$$\hat{k} \cdot \nabla \equiv \cos \vartheta \, \partial_x \qquad (\partial_x = \partial/\partial x). \tag{A.9}$$

Using Eq. (A.9) and making the change of variable from ε to k' (which may be positive or negative) and is defined by

$$\varepsilon = v_F k' \cos \vartheta \tag{A.10}$$

allows Eq. (A.7) to be written as (the prime on k has been neglected)

$$\mathrm{Tr}_3 [f((\mathbf{p}^2 - k_F^2)/2m; x)]$$

$$\approx \pi v_F N(0) \, A \int_{-1}^{1} d(\cos \vartheta) \, |\cos \vartheta| \, \mathrm{tr} \int \frac{dx \, dk}{2\pi} f(v_F \cos \vartheta(k - i\partial_x); x), \quad (A.11)$$

where $A = \int dy \, dz$ is the area of the system in the y, z plane and we have performed the trivial integration over the azimuthal angle in $d\Omega$.

The last step in the linearization is to observe that the x, k integral appearing in Eq. (A.11) is of exactly the form appearing in the right-hand side of Eq. (A.1) with $d = 1$, allowing us to use that equation from right to left and to return to a trace over the operator x and its conjugate momentum p_x. With the introduction of

$$c = v_F \cos \vartheta \tag{A.12}$$

we obtain

$$\mathrm{Tr}_3 [f((\mathbf{p}^2 - k_F^2)/2m; x)] \approx \frac{\pi N(0) A}{v_F} \int_{-v_F}^{v_F} dc \, |c| \, \mathrm{Tr}_1 [f(cp_x; x)]. \tag{A.13}$$

This result constitutes what we mean by a linearisation of the momentum dependence of the kinetic energy.

APPENDIX B: CALCULATION OF THE DENSITY OF STATES FOR THE ORDER PARAMETER $\Delta(x) = \Delta \exp[-2ikx]$

In this appendix we use Eq. (3.2) to calculate the density of states for the order parameter $\Delta(x) = \Delta \exp[-2ikx]$ corresponding to the off-diagonal self-energy

$$U(x) = \Delta \sigma_1 \exp[2ikx\sigma_3] \tag{B.1}$$

and investigate the result in three limits. The density of states for this order parameter is denoted by $\rho_k(E)$.

We start with Eqs. (3.2a) and (3.2b) which read

$$\rho_k(E) = \frac{\pi N(0) A}{v_F} \int_{-v_F}^{v_F} dc \, |c| \, \text{Tr}[\delta(E-h)] \tag{B.2a}$$

$$h = cp\sigma_3 + U(x). \tag{B.2b}$$

It is convenient to perform a unitary transformation on h,

$$\tilde{h} = \exp[ikx\sigma_3] \, h \, \exp[-ikx\sigma_3]$$
$$= cp\sigma_3 - ck + \Delta\sigma_1, \tag{B.3}$$

and for the operator $\tilde{h}$ we impose a periodic boundary condition on its eigenfunctions: $\psi(L) = \psi(0)$. To leading order in L we obtain

$$\text{Tr}[\delta(E-h)] = \text{Tr}[\delta(E-\tilde{h})]$$
$$= \frac{L}{\pi \, |c|} \frac{|E+ck|}{\sqrt{(E+ck)^2 - \Delta^2}} \Theta((E+ck)^2 - \Delta^2). \tag{B.4}$$

On inserting this result in Eq. (B.2a) we obtain, with $V = AL$ the volume of the system,

$$\rho_k(E) = \frac{N(0) V}{v_F k} \{ \text{sgn}(E + v_F k) \sqrt{(E + v_F k)^2 - \Delta^2} \, \Theta((E + v_F k)^2 - \Delta^2)$$
$$- \text{sgn}(E - v_F k) \sqrt{(E - v_F k)^2 - \Delta^2} \, \Theta((E - v_F k)^2 - \Delta^2) \}. \tag{B.5}$$

We take three limits of this result:

(i) $\Delta \to 0$, $k \neq 0$, leading to the normal state result

$$\rho_N(E) = 2N(0) V. \tag{B.6}$$

This result is necessarily independent of k since in the original Hamiltonian, Eq. (B.2), all k dependence drops out when $\Delta = 0$. Note that the k independence of the $\Delta = 0$ limit is not entirely trivial from the viewpoint of the unitarily transformed Hamiltonian, Eq. (B.3), since the k and Δ dependence are apparently separated.

(ii) $\Delta \neq 0$, $k \to 0$, leading to the standard result for a uniform Fermi superfluid:

$$\rho_0(E) = 2N(0) V \frac{|E|}{\sqrt{E^2 - \Delta^2}} \Theta(E^2 - \Delta^2). \tag{B.7}$$

(iii) $\Delta \neq 0$, $k \to \infty$, leading again to the normal density of states given in Eq. (B.6). Evidently, in the large k limit the Fermions see only the spatial average of the rapidly varying order parameter.

APPENDIX C: Approach to the Density of States of a Uniform System as $a \to \infty$

In this appendix we see how the density of states given in Eq. (6.18) approaches the density of states of a spatially uniform system. For the purposes of this appendix we shall set v_F, Δ, and $2N(0)V$ to unity. There are two cases:

(a) $E^2 < 1$. Here, we use the property of the Heaviside function in Eq. (6.18) to write

$$\rho(E) = a\lambda^2 \Theta(\lambda^2 - \mathrm{th}^2(a\lambda)) \int_{a\lambda}^{\mathrm{arc\ tanh}(\lambda)} \frac{du}{u^2} \frac{1}{\sqrt{\lambda^2 - \mathrm{th}^2(u)}} \left[1 - \frac{1}{\lambda^2} \left(1 - \frac{\mathrm{th}(u)}{u} \right) \right] \qquad (C.1)$$

$$\lambda = \sqrt{1 - E^2}. \qquad (C.2)$$

The Heaviside function now outside the integral vanishes if

$$a > \mathrm{arc\ tanh}(\lambda)/\lambda. \qquad (C.3)$$

For $1 \geqslant E > 0$, the ratio $\mathrm{arc\ tanh}(\lambda)/\lambda$ is finite and hence for a fixed energy in this range the density of states will vanish for sufficiently large a. The point $E = 0$, i.e., $\lambda = 1$, where $\mathrm{arc\ tanh}(\lambda)$ diverges, is a singular point of the density of states and the order of limits $E \to 0$, $a \to \infty$ lead to different answers. This does not pose a problem since the density of states usually appears in an integral and there a single point makes a negligible contribution.

We thus see that for $E^2 < \Delta^2$ in the limit $a \to \infty$, the density of states of Eq. (6.18) vanishes, making it identical, in this range of energies, to the density of states of a uniform system $\rho_0(E)$ given in, e.g., Eq. (5.10).

(b) $E^2 > 1$. Here

$$\rho(E) = a\lambda^2 \int_{a\lambda}^{\infty} \frac{du}{u^2} \frac{\Theta(\lambda^2 - \tan^2(u))}{\sqrt{\lambda^2 - \tan^2(u)}} \left[1 + \frac{1}{\lambda^2} \left(1 - \frac{\tan(u)}{u} \right) \right] \qquad (C.4)$$

$$\lambda = \sqrt{E^2 - 1}. \qquad (C.5)$$

Then for an arbitrary function $f(u)$ we have the identity

$$\int_{a\lambda}^{\infty} du\, f(u) \equiv \sum_{n=1}^{\infty} \int_{a\lambda}^{a\lambda + \pi} du\, f(u + (n-1)\pi). \qquad (C.6)$$

Applying it to Eq. (C.5) and noting that $\tan(u + \pi) = \tan(u)$, we have

$$\rho(E) = a\lambda^2 \sum_{n=1}^{\infty} \int_{a\lambda}^{a\lambda + \pi} du \, \frac{1}{[u + (n-1)\pi]^2} \frac{\Theta(\lambda^2 - \tan^2(u))}{\sqrt{\lambda^2 - \tan^2(u)}}$$

$$\times \left[1 + \frac{1}{\lambda^2} \left(1 - \frac{\tan(u)}{[u + (n-1)\pi]} \right) \right]. \tag{C.7}$$

The Euler Maclaurin summation formula [12] can then be applied to the sum. The leading a^{-1} dependence arises from the factor $[u + (n-1)\pi]^{-2}$ in the integrand,

$$\sum_{n=1}^{\infty} \frac{1}{[u + (n-1)\pi]^2} \approx \int_0^{\infty} du \, \frac{1}{[u + (n-1)\pi]^2} = \frac{1}{\pi(u - \pi)}, \tag{C.8}$$

and for large a (or more strictly large $a\lambda$),

$$a\lambda \int_{a\lambda}^{a\lambda + \pi} du \, \frac{1}{\pi(u - \pi)} \cdots \approx \frac{1}{\pi} \int_{a\lambda}^{a\lambda + \pi} du \cdots, \tag{C.9}$$

the corrections to Eqs. (C.8) and (C.9) being of order a^{-1} down, on the leading terms. Using Eqs. (C.8) and (C.9) in Eq. (C.7) yields an integral that is independent of a plus $O(a^{-1})$ corrections. On evaluation, we obtain

$$\rho(E) = \frac{\sqrt{\lambda^2 + 1}}{\lambda} + O(a^{-1}).$$

$$\equiv 2N(0)V \frac{|E|}{\sqrt{E^2 - \Delta^2}} + O(a^{-1}). \tag{C.11}$$

For $a \to \infty$ this is identical to the $E^2 \geqslant \Delta^2$ density of states of a uniform system.

We have therefore shown generally that Eq. (6.18) for the density of states approaches the result of a system with uniform order parameter for large a.

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