Perturbative Approach to Landau-Zener Transitions

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Abstract

The Landau-Zener level-crossing problem is analysed perturbatively.

We consider the operator $H(t) = \Phi(t)\sigma^3 + \Delta\sigma^1$ where $\Phi(t)$ is an external (time-dependent) force, σ^i are the Pauli matrices and Δ is taken to be a small parameter.

By considering the operator $U(t, -t)$ that evolves states in time from $-t$ to t, it is possible to develop a perturbation series in Δ. For three choices of $\Phi(t)$, $U(t, -t)$ is determined for small Δ.

For $\Phi(t)$ periodic with period θ a small Δ approximation to $U(t, -t)$ is obtained that is valid for times of order $\theta/(\theta\Delta)^3$. A slow modulation of this time evolution operator is found and for certain values of the parameters the slow modulation may stop, leaving $U(t, -t) \approx 1$ for all t.

The effects of fluctuations in $\Phi(t)$ on the transition probability are also calculated. To $O\left(\Delta^2\right)$ it is found that for $\Phi(t) \propto t$, white Gaussian noise modifies the probability of a transition in a finite time but leaves unaltered the long time transition probability. This is in accordance with the findings of Y Kayanuma (Phys Rev Lett **58** (1987), 1934).

1 Introduction

When two instantaneous eigenvalues of a quantum mechanical system come close together due to an external time-dependent force[1], transitions between the states associated with the eigenvalues occur. These processes are conventionally referred to as Landau-Zener level-crossing transitions. Such a situation is common in a number of branches of physics as well as biophysics and chemistry and the results of Landau and Zener have applications in these subjects (see [1] for a list of some recent applications).

The purpose of the present work is to analyse the level-crossing problem by perturbation theory. This yields a surprisingly large amount of information on the dynamics of the system and, as we show below, the leading non-trivial terms of perturbation theory require only the evaluation of integrals for their determination.

In Section 2 we give the basic perturbation calculation for the time evolution operator that evolves states from time $-t$ to time t; this object has a particularly simple perturbative development. Next, in Section 3, we look at the results for several different choices of the time-dependent external force; this freedom is a luxury made available only by the restriction of the calculation to the perturbative regime. Of particular interest is the case of an external force that is periodic in time; here an approximate time-evolution operator is found that applies for many cycles of the external force. Section 4 deviates from the path of the previous sections and focuses on the transition probability. It is possible to explicitly include the effects of random fluctuations in the external force and, for a particular choice of external forcing, we determine the transition probability as a function of time. The work is concluded with a short summary and there are two appendices.

All calculations are performed with $\hbar$ set to unity.

2 Perturbative calculation

Zener's original paper on the level-crossing problem [2] was concerned with transitions from the ground state to the excited state of an essentially two level system due to a time-dependent force in the Hamiltonian.

In terms of a pair of basis states which may be taken as $|1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|2\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the Hamiltonian adopted in [2] was equivalent to

$$\begin{aligned} H(t) &= \Phi(t)(|1\rangle\langle 1| - |2\rangle\langle 2|) + \Delta(|1\rangle\langle 2| + |2\rangle\langle 1|) \\ &\equiv \Phi(t)\sigma^3 + \Delta\sigma^1 \end{aligned} \tag{1}$$

where σ^i $(i = 1, 2, 3)$ are the Pauli matrices.

Zener made the choice $\Phi(t) = vt$ with v (and Δ) positive and time-independent. For the purposes of this section we shall, however, consider more general $\Phi(t)$ and specify only that the function is odd:

$$\Phi(-t) = -\Phi(t). \tag{2}$$

If $U(t, t_0)$ (a 2×2 matrix) denotes the time evolution operator for $H(t)$:

[1] We shall use the term "force" to label the time-dependent term in the Hamiltonian originating from an external influence.

$$i\partial_t U(t,t_0) = H(t)U(t,t_0), \qquad U(t_0,t_0) = 1, \tag{3}$$

then in Zener's original calculation in which the ground and first excited states for $t \to -\infty$ were $|1\rangle$ and $|2\rangle$, the asymptotic (long time) transition probability between these states can be cast in the form

$$\lim t \to \infty t_0 \to -\infty |\langle 2|U(t,t_0)|1\rangle|^2 = 1 - \exp(-\pi\Delta^2/v) = \pi\Delta^2/v + \ldots\ . \tag{4}$$

This result is analytic in Δ. It suggests that a perturbative expansion, e.g. of the time evolution operator, in powers of Δ is possible for general t and we shall now proceed to develop this expansion.

We assume that the result in (4) is insensitive to the order in which the limits are taken and the result of this equation follows from the large t limit of the "symmetric" time-evolution operator $U(t,-t)$. We shall study this evolution operator for general positive t and accordingly define[2]:

$$V(t) \equiv U(t,-t), \qquad t \geq 0. \tag{5}$$

The virtue of the evolution operator $V(t)$ is that a perturbation expansion is more naturally and simply expressed in terms of it rather than $U(t,t_0)$. As a simple example of this, consider the case $\Delta = 0$. Since $\Phi(t)$ is odd, $V(t) = 1$ for all times while by contrast, $U(t,t_0)$ is, in general, different from unity (in this case however, $U(t,t_0)$ and $V(t)$ are both diagonal matrices and neither leads to transitions between $|1\rangle$ and $|2\rangle$).

The formal solution for $V(t)$ that follows from (3) is

$$V(t) = T\exp(-i\int_{-t}^{t} H(s)ds), \qquad t \geq 0 \tag{6}$$

where T denotes the usual time ordering operator and is required since, in general, the commutator $[H(t), H(t')] \neq 0$ for $t \neq t'$. Differentiating the above equation with respect to t leads to

$$i\partial_t V(t) = H(t)V(t) + V(t)H(-t)\,. \tag{7}$$

and using (1) and (2) allows (7) to be written as[3]

$$i\partial_t V(t) = [\Phi(t)\sigma^3, V(t)] + \{\Delta\sigma^1, V(t)\}. \tag{8}$$

Let us define

$$\alpha(t) = \int_0^t \Phi(s)\,ds \tag{9}$$

and introduce $W(t)$ via

$$V(t) = e^{-i\alpha(t)\sigma^3} W(t) e^{i\alpha(t)\sigma^3}. \tag{10}$$

This leads to

$$i\partial_t W(t) = \Delta\{e^{2i\alpha(t)\sigma^3}\sigma^1, W(t)\}, \qquad W(0) = 1, \tag{11}$$

[2] In investigating the time evolution of the system, it is possible to look at $U(t,t_0)$ as a function of t when t_0 is held fixed at a finite but arbitrary value. Alternatively we can make the symmetric time choice and consider $U(t,-t)$. Both objects lead to the same long-time ($t \to \infty$, $t_0 \to -\infty$) transition probabilities but differ in their functional dependence on t. Given that each has implications that are experimentally accessible, it is merely a matter of choice and convenience which one is considered and the present paper deals with the symmetric time choice.

[3] In appendix A we provide an alternative formulation of the equation of motion for $V(t)$, Eq.(8), by going to a vector representation.

or

$$W(t) = 1 - \int_0^t ds\{i\Delta e^{2i\alpha(s)\sigma^3}\sigma^1, W(s)\}, \tag{12}$$

A development of $W(t)$ in powers of Δ is obtained by iterating this equation[4]. Even low order terms contain a significant amount of information since (i) $\Phi(t)$ is kept to all orders in each term and (ii) the results apply for arbitrary values of the time t, thereby making some global properties of the time evolution readily accessible.

3 Results for different $\Phi(t)$

Iterating (12) a few times leads, after a little simplification, to[5]

$$\begin{aligned} W(t) &= 1 - \Omega(t) + \frac{\Omega^2(t)}{2} + O((\Delta t)^3), \\ \Omega(t) &= 2i\Delta \textstyle\int_0^t ds\ e^{2i\alpha(s)\sigma^3}\sigma^1. \end{aligned} \tag{13}$$

Thus the perturbative treatment of the time evolution operator rests only on the evaluation of integrals (which can, if necessary, be calculated numerically). Because of this it allows us to consider choices of $\Phi(t)$ that may not lead to an exact solution for the evolution operator in terms of special functions.

We shall evaluate the above expression for $W(t)$ for three different choices of $\Phi(t)$: $\Phi_i(t) \quad (i = 1, 2, 3)$ and quantities associated with the different Φ's will also be labelled accordingly.

3.1 $\Phi_1(t) = vt$

For the above choice for $\Phi(t)$ the corresponding $\alpha(t)$ and $\Omega(t)$ are

$$\alpha_1(t) = \frac{vt^2}{2}, \qquad \Omega_1(t) = 2i\Delta\sqrt{\frac{\pi}{2v}}[C_1(\sqrt{v}t)\sigma^1 - S_1(\sqrt{v}t)\sigma^2] \tag{14}$$

where $C_1(x)$ and $S_1(x)$ are the Fresnel Integrals [3]

$$C_1(x) = \sqrt{\frac{2}{\pi}}\int_0^x du \cos(u^2), \quad S_1(x) = \sqrt{\frac{2}{\pi}}\int_0^x du \sin(u^2). \tag{15}$$

Substituting $\alpha_1(t)$ and $\Omega_1(t)$ into (10) and (13), gives an explicit expression for the symmetric time evolution operator which applies for arbitrary values of t. To linear order in Δ,

$$V_1(t) \equiv U_1(t, -t) = e^{-i\alpha_1(t)\sigma^3}\left[1 - 2i\Delta\sqrt{\frac{\pi}{2v}}[C_1(\sqrt{v}t)\sigma^1 - S_1(\sqrt{v}t)\sigma^2] + ...\right] e^{i\alpha_1(t)\sigma^3} \tag{16}$$

To make connection with the result of Zener for the asymptotic transition probability (4), we use the property of the Fresnel integrals: $\lim_{z\to\infty} C_1(z) = \lim_{z\to\infty} S_1(z) = \frac{1}{2}$. Thus

$$V_1(t) \approx e^{-i\alpha_1(t)\sigma^3}\left[1 - i\Delta\sqrt{\frac{\pi}{2v}}[\sigma^1 - \sigma^2]\right] e^{i\alpha_1(t)\sigma^3}, \qquad t \gg 1/\sqrt{\nu}. \tag{17}$$

[4]Writing $W(t) = \sum_{n=0}^{\infty} W_n(t)$ where $W_0(t) = 1$ and $W_n(t) = O(\Delta^n)$ allows (12) to be written in a form suitable for repeated iteration. With $\omega(s) = -i\Delta e^{2i\alpha(s)\sigma^3}$ we have $W_{n+1}(t) = \int_0^t ds\{\omega(s), W_n(s)\}, \quad n = 1, 2, 3...$.

[5]It may be shown that $\Omega^2(t)$ in (13) is a multiple of the identity matrix. It is therefore a correction to the "1" in $W(t)$ and $(1 + \Omega^2(t)/2)^2 \approx 1 + \Omega^2(t)$ is, to $O(\Delta^2)$, the probability of a "direct transition."

This leads to

$$\lim_{t\to\infty} |\langle 2|U_1(t,-t)|1\rangle|^2 \ \approx \frac{\pi\Delta^2}{v}, \tag{18}$$

reproducing correctly the leading small Δ approximation of Zener's original calculation (4).

3.2 $\Phi_2(t) = V_0 \tanh\left(\frac{t}{\tau}\right)$

With this choice we have

$$\alpha_2(t) = V_0\tau \ln\cosh\left(\frac{t}{\tau}\right), \qquad \Omega_2(t) = 2i\Delta\tau\left(\int_0^{t/\tau} ds \exp\left[2iV_0\tau\sigma^3 \ln\cosh(s)\right]\right)\sigma^1. \tag{19}$$

Provided we include a convergence factor into the integrand of $\Omega_2(t)$ (which for convenience we take to be $\exp[-\delta\ \ln\cosh(s)]$, with δ a positive infinitesimal) then $\Omega_2(t)$ converges in the limit $t\to\infty$ (see [5], Eq.(3.512 2)) and

$$\lim_{t\to\infty} W_2(t) \approx 1 - i\Delta\tau\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-i\sigma^3V_0\tau\right)}{\Gamma\left(\frac{1}{2}-i\sigma^3V_0\tau\right)}\sigma^1. \tag{20}$$

We can justify the inclusion of the convergence factor by comparing the implications of (20) with the small Δ approximation to the exact result for, e.g., $\lim_{t\to\infty}|\langle 2|U_2(t,-t)|1\rangle|^2$. Both (20) and the exact result lead to

$$\lim_{t\to\infty} |\langle 2|U_2(t,-t)|1\rangle|^2 \approx \frac{\Delta^2\pi\tau}{V_0}\coth(\pi V_0\tau) \tag{21}$$

provided $\Delta^2 \ll V_0^2$ and also $\left(\Delta^2\pi\tau/V_0\right)\coth(\pi V_0\tau) \ll 1$. Note that (18) can be obtained from (21) by writing $V_0 = v\tau$ and taking the limit $\tau\to\infty$. Note also that the limit $\tau\to 0$ of (21) leads to the small Δ approximation for the "sudden" variation $\Phi(t) = sgn(t)$.

3.3 $\Phi_3(t) = A\sin(2\pi t/\theta)$

In this case it is advantageous to shift $\alpha(t)$ by an additive constant relative to the definition in (9):

$$\alpha_3(t) \equiv \int_0^t \Phi_3(s)\,ds + \frac{A\theta}{2\pi} = -\frac{A\theta}{2\pi}\cos\left(\frac{2\pi t}{\theta}\right). \tag{22}$$

All other expressions depending on α remain unaltered in form, thus

$$\Omega_3(t) = 2i\Delta\int_0^t ds\ \exp\left[-i\frac{A\theta}{\pi}\cos\left(\frac{2\pi s}{\theta}\right)\sigma^3\right]\sigma^1. \tag{23}$$

$\Phi_3(t)$ is periodic over the interval θ and we shall consider the evolution operator for $t=\theta/2$. Since $V(t)$ evolves states from $-t$ to t, $V_3(\theta/2)$ corresponds to evolution over a one *complete* period of $\Phi_3(t)$.

The result for $V_3(\theta/2)$ is found, in linear order, to be

$$V_3(\theta/2) \equiv U(\theta/2,-\theta/2) = e^{-i\alpha_3(\theta/2)\sigma^3}\left[1 - i\Delta\,\theta\ J_0\left(\frac{A\theta}{\pi}\right)\sigma^1 + ...\right]e^{i\alpha_3(\theta/2)\sigma^3} \tag{24}$$

where $J_0(z)$ is a Bessel function of the first kind of order zero [3] and we assume "small Δ" means here: $\theta\Delta \ll 1$.

Equation (24) can be used to determine the time evolution operator over an integer number of periods of $\Phi_3(t)$. Using the results of Appendix A, we have

$$V_3\left(n\frac{\theta}{2}\right) = \begin{cases} [V_3(\theta/2)]^n & n \text{ even} \\ \sigma^1 [V_3(\theta/2)]^n \sigma^1 & n \text{ odd} \end{cases} \tag{25}$$

Using $\left[1 - \Omega(\theta/2) + \Omega^2(\theta/2)/2 + O((\theta\Delta)^3)\right]^n = \exp[-n\Omega(\theta/2) + O(n(\theta\Delta)^3)]$, we find we can write the small Δ approximation of $V_3(n\theta/2)$ (for both even and odd n) as

$$V_3(n\theta/2) \approx e^{-i\alpha_3(n\theta/2)\sigma^3} \exp\left[-i\,(n\theta)\,\Delta J_0\left(\frac{A\theta}{\pi}\right)\sigma^1\right] e^{i\alpha_3(n\theta/2)\sigma^3}, \quad n = 1, 2, 3... \tag{26}$$

corrections being of relative order $n(\theta\Delta)^3$. Because of our assumption that $\theta\Delta \ll 1$, the corrections become significant only after many cycles of $\Phi_3(t)$, (i.e. $n \sim (\theta\Delta)^{-3} \gg 1$).

$V(n\theta/2)$ evolves states through a time of $n\theta$ and (26) indicates, by the inner exponential, that there is a unit amplitude, slow frequency modulation underlying the time evolution. Thus, on top of the comparatively rapid oscillations resulting from the $e^{\pm i\alpha(n\theta/2)\sigma^3}$ factors, there is a slow modulation with period

$$\Theta = \frac{2\pi}{\Delta\left|J_0\left(\frac{A\theta}{\pi}\right)\right|}. \tag{27}$$

The period, Θ, is a non-linear function of the parameters in the problem and when $A\theta/\pi$ coincides with a zero of the Bessel function, Θ can become infinite. When this occurs, it indicates that to the accuracy we are working to, $V_3(n\theta/2) = 1$ for all integer n. Thus for a particular combination of parameters, the periodic forcing results in essentially no transitions between $|1\rangle$ and $|2\rangle$. This feature is not unique to a sinusoidal $\Phi(t)$; a number of different choices of *periodic* $\Phi(t)$ all yield the same behaviour. It appears the divergence of Θ (vanishing of the frequency of slow modulation) for a particular combination of parameters is a general feature of periodic forcing. A possibly related behaviour has been seen in a different context involving two-level systems associated with circular motion [6].

We end this section by noting that (26) suggests an approximation to $V_3(t)$ for the wide range of times satisfying $t \ll \theta\ (\theta\Delta)^{-3}$, namely exponentiate $\Omega(t)$ in (13):

$$V_3(t) \equiv U(t,-t) \approx e^{-i\alpha_3(t)\sigma^3} e^{\left[-2i\Delta\int_0^t ds\ e^{2i\alpha_3(s)\sigma^3}\sigma^1\right]} e^{i\alpha_3(t)\sigma^3}. \tag{28}$$

4 Effect of fluctuations

So far we have considered a two-level system forced by a function $\Phi(t)$ which is odd in t. It commonly happens that superimposed on the systematic forcing are random fluctuations $f(t)$. We make the assumption that the fluctuations are completely characterized by a distribution function ("classical fluctuations") and shall average observable physical quantities over $f(t)$ to yield ensemble averages. A recent example of such fluctuations in the context of population inversion in lasers is given in work by Kayanuma[6] [4]; we shall not deal here with the fluctuations associated with quantum environments, (see for example [1]).

[6]It may be shown that the equation of motion of the density matrix appearing in Eq.(3) of [4] follows (in the notation of Ref.[4]) from the Hamiltonian $\left(\frac{vt}{2} + f(t)\right)(|1\rangle\langle 1| - |2\rangle\langle 2|) + J(|1\rangle\langle 2| + |2\rangle\langle 1|)$ where $f(t)$ is a delta correlated Gaussian random variable with zero mean. On averaging the equation of motion resulting from the above Hamiltonian with respect to $f(t)$, Eq.(3) of [4] results.

To proceed it is necessary to generalise the treatment given in Section 2 to the case where $\Phi(t)$ is *not* an odd function of t. With the introduction of

$$\alpha_{\pm}(t) = \int_0^t ds\, \Phi(\pm s) \tag{29}$$

and following the procedure of Section 2, we find that to linear order in Δ:

$$W(t) = 1 - i\Delta \int_0^t ds \left[\exp\left(2i\alpha_+(s)\sigma^3\right) + \exp\left(-2i\alpha_-(s)\sigma^3\right)\right]\sigma^1. \tag{30}$$

Next we take

$$\Phi(t) = \Phi_0(t) + f(t) \tag{31}$$

where $\Phi_0(t)$ is deterministic and odd:

$$\Phi_0(-t) = -\Phi_0(t) \tag{32}$$

and $f(t)$ is random variable. We take $f(t)$ to have a Gaussian white noise distribution (an over-bar denotes an ensemble average with respect to $f(t)$) characterized by

$$\overline{f(t)} = 0, \quad \overline{f(t_1)f(t_2)} = \lambda\,\delta(t_1 - t_2). \tag{33}$$

Then with

$$\alpha_0(t) = \int_0^t ds\, \Phi_0(s), \quad \beta_{\pm}(t) = \int_0^t ds\, f(\pm s) \tag{34}$$

(30) takes the form

$$W(t) = 1 - i\Delta \int_0^t ds\, \exp(2i\alpha_0(s)\sigma^3)\left[\exp(2i\beta_+(s)\sigma^3) + \exp(-2i\beta_-(s)\sigma^3)\right]\sigma^1. \tag{35}$$

It is inappropriate to average $U(t,-t) \equiv V(t)$ (given in (10)) with respect to $f(t)$ since this contains probability *amplitudes* and is not an observable quantity. Instead transition *probabilities* should be averaged. We therefore consider the average of

$$\begin{aligned} P(t) &\equiv |\langle 2|U(t,-t)|1\rangle|^2 \qquad (36)\\ &= |\langle 2|V(t)|1\rangle|^2 = |\langle 2|W(t)|1\rangle|^2. \end{aligned}$$

Omitting some straightforward algebra, it follows from (35) and the properties of the Gaussian noise that to lowest non-trivial order in Δ,

$$\begin{aligned} \overline{P(t)} &= 2\Delta^2 \textstyle\int_0^t ds_1 \int_0^t ds_2\, e^{2i[\alpha_0(s_1)-\alpha_0(s_2)]}\left[e^{-2\lambda|s_1-s_2|} + e^{-2\lambda(s_1+s_2)}\right] \\ &= 4\Delta^2 \textstyle\int_0^t ds_1 \int_0^{s_1} ds_2\, \cos\left(2\left[\alpha_0(s_1) - \alpha_0(s_2)\right]\right)\left[e^{-2\lambda(s_1-s_2)} + e^{-2\lambda(s_1+s_2)}\right]. \end{aligned} \tag{37}$$

We shall evaluate this expression for the special case where

$$\Phi_0(t) = vt, \quad \alpha_0(t) = \frac{vt^2}{2}. \tag{38}$$

We change variables in (37) to $x = s_1 - s_2,\ y = s_1 + s_2$, and symmetry of the integrand with respect to $x \longleftrightarrow y$ allows us to enlarge the domain of integration in the (x, y) plane to the triangular region bounded by the x and y axes and the line $x + y = 2t$:

$$\begin{aligned}\overline{P(t)} &= \Delta^2 \int_0^{2t} dx \int_0^{2t} dy\, \Theta(2t - x - y) \cos(vxy) \left[e^{-2\lambda x} + e^{-2\lambda y}\right] \\ &= 2\Delta^2 \int_0^{2t} dx \int_0^{2t-x} dy\, e^{-2\lambda x} \cos(vxy)\end{aligned} \tag{39}$$

carrying out the y-integration and rescaling the results leads to

$$\overline{P(t)} = \frac{\pi\Delta^2}{v} I(vt^2; \frac{\lambda}{vt}), \quad I(a, b) \equiv \frac{2}{\pi} \int_0^{4a} \frac{du}{u} e^{-bu} \sin(u - \frac{u^2}{4a}). \tag{40}$$

Note that

$$\lim_{t\to\infty} I(vt^2; \frac{\lambda}{vt}) = 1 \tag{41}$$

so to $O(\Delta^2)$, $\overline{P(\infty)} = \dfrac{\pi\Delta^2}{v}$ i.e. independent of λ and hence the noise[7]. It follows from (40) however, that for all finite values of t, $\overline{P(t)}$ does depend on t. In the Figure we present plots of $\overline{P(t)}/\overline{P(\infty)} \equiv I(vt^2; \frac{\lambda}{vt})$ as a function of t for λ zero and non-zero. It is evident that the effect of the noise is to accelerate convergence of the probability to its large t limit.

5 Summary

In this work we have explored Zener tunnelling in the perturbative regime corresponding to small Δ. It has been possible to determine the time evolution operator for a number of different choices of the forcing term $\Phi(t)$ including the interesting case where it is periodic. It has also been possible to determine the effects of fluctuations on the transition probability.

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[7]The results of [4], when expanded to *leading* non-zero order in Δ, are independent of the noise. Beyond leading order, however, the results of this reference show that $\overline{P(\infty)}$ does depend on the noise.

A Vector representation for the symmetric evolution operator $V(t)$

In this appendix we express the symmetric evolution operator, $V(t) = U(t, -t)$ in terms of a vector representation. Such a formulation may shed insight into the behaviour of $V(t)$.

We proceed by noting the following implication of the equation of motion for $V(t)$, Eq.(8), combined with the initial condition, $V(0) = 1$, namely, $V(t)$ consists of a combination of σ^0 (the unit 2×2 matrix), σ^1 and σ^2, but is independent of σ^3. This suggests the vector representation:

$$V(t) = (\mathbf{a}.\boldsymbol{\sigma})\sigma^3, \qquad \mathbf{a} = (a_x, a_y, a_z). \tag{42}$$

Substituting this into Eq.(8) yields

$$\frac{d\mathbf{a}}{dt} = \mathbf{b} \wedge \mathbf{a}, \qquad \mathbf{b}(t) = (2\Delta, 0, 2\Phi(t)). \tag{43}$$

The factors of 2 appearing in (43) arise from the fact that $V(t)$ evolves over a time interval of $2t$.

The initial condition $V(0) = 1$ implies $\mathbf{a}(0) = (0, 0, 1)$. Reality of $\mathbf{b}(t)$ and Eq.(43) then ensures that $\mathbf{a}(t)$ remains real and of unit length for all times.

B Some properties of the time evolution operator for periodic Hamiltonians

In this appendix we derive some properties of the time evolution operator that apply when the external force is periodic over the interval θ and has the properties

$$\Phi(t+\theta) = \Phi(\theta) \tag{44}$$

$$\Phi(t+\theta/2) = -\Phi(\theta). \tag{45}$$

An example of such a function is $\Phi(t) = A\sin(\frac{2\pi t}{\theta})$.

B.1 Translation through a full period

First, the Hamiltonian $H(t)$ (1) is periodic

$$H(t+\theta) = H(t), \quad \forall t \tag{46}$$

and by shifting both t and t_0 by θ in (3) we obtain $[i\partial_t - H(t)]\, U(t+\theta, t_0+\theta) = 0$. Then $U(t,t) = 1$, results in $U(t_0+\theta, t_0+\theta) = 1$, and since $U(t+\theta, t_0+\theta)$ obeys the same differential equation and boundary condition as $U(t, t_0)$ it follows that

$$U(t+\theta, t'+\theta) = U(t, t'). \tag{47}$$

B.2 Translation through a half period

Let us shift both t and t_0 by $\theta/2$ in (3). Since $\Phi(t+\theta/2) = -\Phi(t)$, (3) leads to

$$\left(i\partial_t - \left[-\Phi(t)\sigma^3 + \Delta\sigma^1\right]\right) U(t+\theta/2, t_0+\theta/2) = 0, \quad U(t_0+\theta/2, t_0+\theta/2) = 1. \tag{48}$$

By pre and post multiplying this equation by σ^1 and using the properties of the Pauli matrices we obtain

$$[i\partial_t - H(t)]\, \sigma^1 U(t+\theta/2, t_0+\theta/2)\sigma^1 = 0,\ \sigma^1 U(t_0+\theta/2, t_0+\theta/2)\sigma^1 = 1. \tag{49}$$

Again making the comparison with $U(t, t_0)$ we infer

$$\sigma^1 U(t+\theta/2, t'+\theta/2)\sigma^1 = U(t, t'). \tag{50}$$

B.3 Expressing $U(n\theta/2, -n\theta/2)$ in terms of $U(\theta/2, -\theta/2)$.

We can combine the results of (47) and (50) with the group property

$$U(t, t'')U(t'', t') = U(t, t') \tag{51}$$

to express $U(n\theta/2, -n\theta/2)$, with n integral, in terms of $U(\theta/2, -\theta/2)$.

We have, by repeated application of (47):

$$U(n\theta/2, -n\theta/2) = \begin{cases} [U(\theta/2, -\theta/2)]^n & n \text{ odd} \\ [U(\theta, -\theta)]^{n/2} & n \text{ even.} \end{cases} \tag{52}$$

This can be simplified. Application of (51), (47) and (50) yields

$$U(\theta,-\theta)=U(\theta,0)U(0,-\theta)=U(\theta,0)U(\theta,0)=\left[\sigma^1 U(\theta/2,-\theta/2)\sigma^1\right]^2 \tag{53}$$

thus (52) can be written

$$U(n\theta/2,-n\theta/2)=\begin{cases}\left[U(\theta/2,-\theta/2)\right]^n & n \text{ odd}\\ \sigma^1\left[U(\theta/2,-\theta/2)\right]^n\sigma^1 & n \text{ even}\end{cases} \tag{54}$$

and this is the principal result of this appendix.

References

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Figure Caption

We define $\overline{P(t)} = \overline{|\langle 2|U(t,-t)|1\rangle|^2} \equiv$ the noise-averaged transition probability. With $\overline{P(t)}$ and $\overline{P(\infty)}$ calculated to $O(\Delta^2)$, the ratio $\overline{P(t)}/\overline{P(\infty)}$, with is plotted as a function of t. The deterministic part of the external force has been taken as $\Phi_0(t) = vt$ with $v = 1$ throughout. The two graphs correspond to the noise autocorrelation parameter λ (see equation (33))taking the values 0 and $\frac{1}{2}$. We note that to $O(\Delta^2)$, $\overline{P(\infty)}$ is independent of λ but $\overline{P(t)}$, for finite t, does depend on λ.