J. Phys. A: Math. Gen. 21 (1988) 3677-3683. Printed in the UK

Instanton corrections to the Maxwell construction—a simple example

D Waxman

Physics Division, University of Sussex, Brighton, BN1 9QH, Sussex, UK

Received 21 January 1988

Abstract. A zero-dimensional field theory is studied in which the potential appearing in the Lagrangian is non-convex. By using a variant of the instanton approximation the effective potential is obtained. This is found to be convex with a curvature of order $e^{-1/\hbar}$. This is due to non-perturbative (tunnelling) processes. The Maxwell construction is retrieved in the $\hbar \to 0$ limit.

1. Introduction

The effective potential in field theory is a function whose minimum lies at the ground-state expectation value of the field. It has a precise analogue in statistical mechanics, namely the free energy expressed as a function of the magnetisation (or order parameter). We shall utilise the language of field theory for this paper—we could equally have used that of statistical mechanics.

The usual objective is to find the effective potential as a function of the static homogeneous field. This is often achieved only at the expense of making approximations—many of which lead to a non-convex function of the field. Elementary arguments [1] show that the effective potential must be convex. The *ad hoc* procedure which is employed to rid the effective potential of its non-convex portions is the so-called Maxwell construction. In this the convex cover (or convex hull) of the approximate effective potential is taken.

It seems natural to enquire into the validity of the Maxwell construction. For instance, does the exact effective potential† have a strictly flat region or does it have some curvature?

In this paper we consider a system whose effective potential can be found to sufficient accuracy that:

(i) deviations from the Maxwell construction can be seen and their origin understood;

(ii) the Maxwell construction is seen to follow naturally in an appropriate limit.

This paper is arranged as follows. In § 2 we set up notation and define the effective potential and the system under study. In § 3 we utilise a variant of the instanton approximation to obtain the effective potential. This section is supplemented by an appendix which obtains the results in an alternative way. The paper is concluded with § 4.

† In some cases (e.g., gauge theories) the effective potential is not unique but gauge dependent. In the following work we consider a system with a unique effective potential. I thank D Bailin for informing me of this point.

2. Defining the system and the effective potential

In order to achieve the aims of this paper we have to consider the simplest quantum field theory we know of: a scalar field in no space and one time. This, of course, coincides with the quantum mechanics of a single particle.

The Euclidean Lagrangian is

$$\mathscr{L}=\tfrac{1}{2}\dot{\phi}^2+V(\phi) \tag{1}$$

where $\dot{\phi}\equiv \mathrm{d}\phi/\mathrm{d}t$ and $V(\phi)$ has the double well form given in figure 1.

We define the function $F(J)$ and the generating function $Z(J)$ via†

$$Z(J)=\mathrm{e}^{-\tau F(J)/\hbar}=\int \mathrm{d}[\phi]\exp\left[-\frac{1}{\hbar}\left(\int_0^\tau \mathscr{L}\,\mathrm{d}t-J\int_0^\tau \phi\,\mathrm{d}t\right)\right] \tag{2}$$

with τ the size of the Euclidean time region. Strictly speaking, the form $\mathrm{e}^{-\tau F(J)/\hbar}$ only holds as $\tau\to\infty$. This is the limit we are interested in and we shall take this limit at the end of the calculation for $F(J)$.

The classical or static field $\bar{\phi}(J)$ is defined by

$$\bar{\phi}(J)\equiv\left\{\int \mathrm{d}[\phi]\exp\left[-\frac{1}{\hbar}\left(\int_0^\tau \mathscr{L}\,\mathrm{d}t-J\int_0^\tau \phi\,\mathrm{d}t\right)\right]\left(\frac{1}{\tau}\int_0^\tau \phi\,\mathrm{d}t\right)\right\}(Z(J))^{-1} \tag{3}$$

$$=\frac{\hbar}{\tau}\frac{\partial}{\partial J}\ln Z(J) \tag{4}$$

i.e.

$$\bar{\phi}(J)=-\frac{\partial F(J)}{\partial J}. \tag{5}$$

Assuming that $\bar{\phi}(J)$ is a monotonic function‡ of J we can invert (5) to obtain

$$J=J(\bar{\phi}). \tag{6}$$

The effective potential $U(\bar{\phi})$ is defined as the Legendre transform

$$U(\bar{\phi})=F(J)+\bar{\phi}J \tag{7}$$

with J expressed as a function of $\bar{\phi}$.

It follows from (7) and (5) that

$$\frac{\mathrm{d}U(\bar{\phi})}{\mathrm{d}\bar{\phi}}=J \tag{8}$$

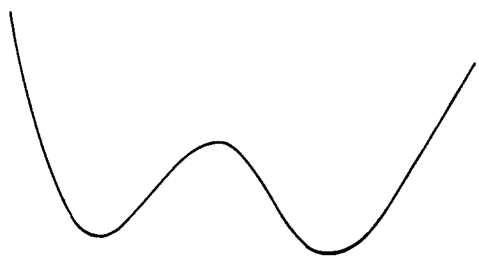

Figure 1. The double well potential appearing in the Lagrangian.

† $\int \mathrm{d}[\phi]\ldots$ denotes a functional integral over ϕ.

‡ This is explictly verified in (21).

and thus when $J=0$, $\mathrm{d}U/\mathrm{d}\bar{\phi}=0$ yields the ground-state expectation value of $\bar{\phi}$. It is a simple matter to show that $\partial^2 U/\partial\bar{\phi}^2 \geqslant 0$ (see, e.g., [1]) and hence the effective potential is convex and its minimum coincides with the ground-state value of $\bar{\phi}$.

We shall now find the effective potential in an appropriate approximation.

3. Instanton approximation

We have

$$\mathrm{e}^{-F(J)\tau/\hbar} = \int \mathrm{d}[\phi] \exp\left(-\frac{1}{\hbar}\int_0^\tau \mathrm{d}t[\tfrac{1}{2}\dot{\phi} + V(\phi) - J\phi]\right) \tag{9}$$

with $F(J)$ determined in the $\tau\to\infty$ limit.

If J is sufficiently small (a definite criterion will be given later) it is reasonable to assume that the dominant trajectories in the functional integral take the form of those given in figure 2. These are composed of instantons† and anti-instantons (see, e.g., [2]). We approximate the functional integral by summing over all such trajectories. In the approximation in which we are working we take the instanton profile to be determined by the J-independent part of the Euclidean action‡ (and therefore the anti-instanton is the mirror image of the instanton).

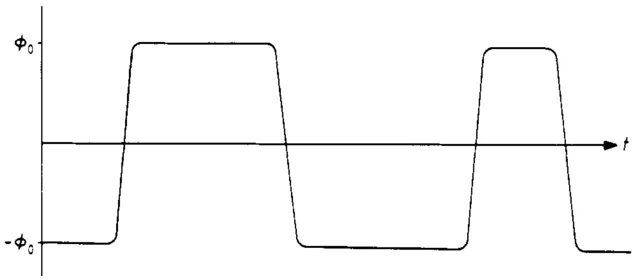

Figure 2. A trajectory of the multi-instanton type used in the path integral.

For $J=0$ we denote the Euclidean action of a single instanton by S_0 and hence the action of n instantons is nS_0. Consider now a trajectory composed of $2n$ objects: an instanton at 'position' t_1, followed by an anti-instanton at position $t_2, \ldots$. If $J\neq 0$, the action of this trajectory, S_{2n}, is given by

$$S_{2n}(J) = 2nS_0 - J\int_0^\tau \phi(t)\,\mathrm{d}t \tag{10}$$

$$\simeq 2nS_0 - J\left(\int_0^{t_1}(-\phi_0)\,\mathrm{d}t + \int_{t_1}^{t_2}\phi_0\,\mathrm{d}t + \ldots\right) \tag{11}$$

$$= 2nS_0 + J\phi_0 \sum_{j=0}^{2n}(-1)^j(t_{j+1}-t_j) \tag{12}$$

† The instanton is a fast hop from $-\phi_0$ to $+\phi_0$, the anti-instanton is the reversed path—they are a manifestation of quantum tunnelling between the two minima of the potential.

‡ We therefore solve the Euler Lagrange equation of motion to obtain this profile. The procedure of using only the J-independent part of the action was used in [3].

with

$$t_0 \equiv 0, \ t_{2n+1} \equiv \tau. \tag{13}$$

We shall evaluate $Z(J)$ by summing over all trajectories of the above kind (i.e. those which start at $-\phi_0$ and end at $-\phi_0$)†. In performing the sum we must, of course, include contributions from instantons and anti-instantons at all possible locations. This involves integrating over their positions. We thus obtain (see, e.g., [3] for related calculations)

$$Z(J) = Z_0 \sum_{n=0}^{\infty} \int_0^{\tau} \frac{dt_{2n}}{A} \int_0^{t_{2n}} \frac{dt_{2n-1}}{A} \dots \int_0^{t_2} \frac{dt_1}{A} \exp(-S_{2n}(J)/\hbar). \tag{14}$$

In this equation Z_0 is the generating function when $J=0$ in the absence of tunnelling between the potential minima (i.e. no instanton contributions). We take it to be the free field (i.e. oscillator) result

$$Z_0 = \left(2 \sinh \frac{\omega\tau}{2}\right)^{-1} \tag{15}$$

with ω the small oscillation frequency about ϕ_0

$$\omega = [V''(\phi_0)]^{1/2}. \tag{16}$$

The quantity A appearing in (14) (with the dimensions of time) is given by the ratio of two functional determinants [3].

$Z(J)$ may be found by substituting $S_{2n}(J)$ from (12) into (14) and evaluating the sum. Details of this are given in the appendix. The result is ((A8)-(A10))

$$Z(J) = Z_0 \left(\cosh[(\alpha^2+\beta^2)^{1/2}\tau] - \frac{\beta}{(\alpha^2+\beta^2)^{1/2}} \sinh[(\alpha^2+\beta^2)^{1/2}\tau] \right) \tag{17}$$

with

$$\alpha = \frac{e^{-S_0/\hbar}}{A} \qquad \beta = \frac{J\phi_0}{\hbar}. \tag{18}$$

It is now a simple matter to take the large-τ limit of (17) and hence find $F(J)$:

$$F(J) = \frac{\hbar\omega}{2} - [\Delta^2 + (J\phi_0)^2]^{1/2} \tag{19}$$

where for convenience we have defined

$$\Delta \equiv \frac{\hbar\, e^{-S_0/\hbar}}{A}. \tag{20}$$

The static field $\bar{\phi}(J)$ (5) is thus

$$\bar{\phi}(J) = -\frac{\partial F}{\partial J} = \frac{J\phi_0^2}{[\Delta^2 + (J\phi_0)^2]^{1/2}} \tag{21}$$

which is a monotonic function of J as assumed in (6). Inverting (21) yields

$$J = J(\bar{\phi}) = \frac{\Delta}{\phi_0} \frac{\bar{\phi}/\phi_0}{[1-(\bar{\phi}/\phi_0)^2]^{1/2}}. \tag{22}$$

† There are four sectors of trajectories, corresponding to starting and ending at $\pm\phi_0$. The $\tau \to \infty$ result we obtain for $F(J)$ is unaltered if we include contributions from trajectories of the other three sectors.

This equation appears to allow us to give the criterion of what constitutes small J. It is clear that the unphysical singularities at $\bar{\phi} = \pm\phi_0$ are artefacts of the small-J approximations made. We might expect that $\bar{\phi}$ should be far from these singularities; in which case

$$|J| \leqslant \Delta/\phi_0. \tag{23}$$

However, this is probably too severe. A more realistic criterion would follow from considerations of when levels start crossing in the potential wells†. This suggests

$$|J| \ll \hbar\omega/\phi_0 \tag{24}$$

and this will be taken as the criterion for small J.

We conclude this section by obtaining the quantity at the centre of this paper: the effective potential. Putting the results of (19) and (22) together we obtain

$$U(\bar{\phi}) = F(J) + \bar{\phi}J \tag{25}$$

$$= \frac{\hbar\omega}{2} - \Delta\left[1 - \left(\frac{\bar{\phi}}{\phi_0}\right)^2\right]^{1/2} \tag{26}$$

$$= \frac{\hbar\omega}{2} - \Delta + \frac{\Delta}{2\phi_0^2}\bar{\phi}^2 + \ldots. \tag{27}$$

This is convex (as it must be) with a curvature (at $\bar{\phi} = 0$) of $\Delta/\phi_0^2 \propto e^{-S_0/\hbar}$. Any expansion of the effective potential in powers of $\hbar$ would miss this tiny curvature and lead to an effective potential that is flat between $-\phi_0$ and $+\phi_0$. In this way we retrieve the Maxwell construction.

4. Conclusion

In this work we have provided an example which shows how non-perturbative (instanton) effects yield deviations from the Maxell construction. The low-dimensional nature of the system studied precludes the existence of broken symmetry solutions (i.e. multiple phases). It appears that in systems that are capable of having more than one phase, non-trivial complexities of the effective potential exist [4].

The instanton method has been used in this paper to obtain the effective potential despite the existence of a much simpler method (see the appendix). This was done in the hope that related methods may be used in higher-dimensional theories.

Appendix

In this appendix we evaluate the sum appearing in (14) and provide an alternative derivation of $Z(J)$.

† Therefore the two-level approximation breaks down (see the appendix).

A1. Evaluation of the sum

Equations (14) and (12) lead to

$$Z(J)=Z_0\sum_{n=0}^{\infty}\int_0^{\tau}\frac{\mathrm{d}t_{2n}}{A}\int_0^{t_{2n}}\frac{\mathrm{d}t_{2n-1}}{A}\dots$$
$$\int_0^{t_2}\frac{\mathrm{d}t_1}{A}\exp(-2nS_0/\hbar)\exp\left(-J(\phi_0/\hbar)\sum(-1)^j(t_{j+1}-t_j)\right). \tag{A1}$$

We are thus led to consider

$$I_e=\sum_{n=0}^{\infty}\alpha^{2n}\int_0^{\tau}\mathrm{d}t_{2n}\int_0^{t_{2n}}\mathrm{d}t_{2n-1}\dots\exp\left(-\beta\sum^{2n}(-1)^j(t_{j+1}-t_j)\right) \tag{A2}$$

and

$$I_0=\sum_{n=0}^{\infty}\alpha^{2n+1}\int_0^{\tau}\mathrm{d}t_{2n+1}\int_0^{t_{2n+1}}\mathrm{d}t_{2n}\dots\exp\left(-\beta\sum^{2n+1}(-1)^j(t_{j+1}-t_j)\right) \tag{A3}$$

where in (A2) $t_{2n+1}\equiv\tau$ and in (A3) $t_{2n+2}\equiv\tau$.

It therefore follows that

$$\frac{\partial}{\partial\tau}I_e=\alpha I_0-\beta I_e \tag{A4}$$

$$\frac{\partial}{\partial\tau}I_0=\alpha I_e+\beta I_0. \tag{A5}$$

The solution to these equations subject to

$$I_e(\tau=0)=1 \qquad I_0(\tau=0)=0 \tag{A6}$$

is

$$\begin{pmatrix}I_0(\tau)\\ I_e(\tau)\end{pmatrix}=\exp[(\alpha\sigma^1+\beta\sigma^3)\tau]\begin{pmatrix}0\\ 1\end{pmatrix} \tag{A7}$$

where σ^j denotes the jth Pauli matrix.

Thus

$$I_e(\tau)=\cosh[(\alpha^2+\beta^2)^{1/2}\tau]-\frac{\beta}{(\alpha^2+\beta^2)^{1/2}}\sinh[(\alpha^2+\beta^2)^{1/2}\tau] \tag{A8}$$

and

$$Z(J)=Z_0I_e(\tau) \tag{A9}$$

with

$$\alpha\equiv\frac{\exp(-S_0/\hbar)}{A} \qquad \beta\equiv\frac{J\phi_0}{\hbar}. \tag{A10}$$

A2. Alternative derivation of Z(J)

The result we have obtained for $Z(J)$ may be obtained without using instantons. The derivation rests on the fact that for large τ the only states making a significant contribution to $Z(J)$ correspond to the lowest two states in the double well potential. For small J we may take a basis in this two-dimensional Hilbert space to be $|L\rangle$ and $|R\rangle$ (corresponding to wavefunctions localised about $-\phi_0$ and $+\phi_0$ respectively).

The Hamiltonian consists of the three terms: a zero-point energy term, a tunnelling term and a term arising from the source J.

In a basis in which $|L\rangle$ is represented by $\binom{0}{1}$, $|R\rangle$ by $\binom{1}{0}$ we have (σ^i are the Pauli spin matrices)

$$H \simeq \frac{\hbar\omega}{2} - \Delta\sigma^1 - J\phi_0\sigma^3 \tag{A11}$$

in which Δ is the tunnel matrix element.

If we identify Δ with $\hbar\alpha$ then we find that to corrections of order $e^{-\omega\tau}$, $Z(J)$ and $(0\ \ 1)\, e^{-H\tau/\hbar}\binom{0}{1}$ coincide.

Acknowledgments

I would like to thank D Bailin for reading a draft of the manuscript and providing some helpful comments.

References

[1] Rivers R J 1984 *Z. Phys.* C **22** 137
[2] Coleman S 1979 *The Whys of Subnuclear Physics (Erice, 1977)* ed A Zichichi (New York: Plenum)
[3] Chakravarty S 1982 *Phys. Rev. Lett.* **49** 681
[4] Norimatsu S, Yamamoto K and Tanaka A 1987 *Phys. Rev.* D **35** 2009