The Fredholm Determinant for a Dirac Hamiltonian with a Topological Mass Term

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November 1, 1996

Abstract

We consider the Fredholm determinant associated with two Hamiltonians H and H_0. If these have discrete spectra with eigenvalues E_n and E_{n0} respectively, then the Fredholm determinant, as a function of a complex variable z, is $\mathrm{Det}\left[(z-H)/(z-H_0)\right] \equiv \prod_n \left[(z-E_n)/(z-E_{n0})\right]$. This object contains information on the spectra of the operators H and H_0 which we take to be Dirac Hamiltonians appropriate to one spatial dimension. The Pauli matrices $\widehat{\sigma}^i$ $(i=1,2,3)$ are used as a representation of the Dirac gamma matrices.

An expression for the Fredholm determinant is derived when the term in H involving the mass has a variation with position, x, of the form $\widehat{\Delta}(x) \equiv \Delta_2(x)\widehat{\sigma}^2 + \Delta_3(x)\widehat{\sigma}^3$. We consider the general case $\widehat{\Delta}(-\infty) \neq \widehat{\Delta}(\infty)$ and when this holds, the Hamiltonian is said to possess a topological mass term. The mass term in H_0 is taken to have the same asymptotic limits as $\widehat{\Delta}(x)$ but different local behaviour. We find that the Fredholm determinant can be compactly expressed in terms of the 2×2 matrices characterizing the asymptotic spatial properties of the Green's function $1/(z-H)$.

There are applications of this work to systems involving Fermions coupled to topological solitons. Examples of these have been studied in quantum field theory and condensed-matter physics. In the latter case, a Dirac-like equation may arise as an approximate description of non-relativistic Fermions; the mass term usually having the interpretation as an order-parameter field.

1. Introduction

The Fredholm determinant associated with two first-quantized Hamiltonians, H and H_0 is[1]

$$\left.\begin{aligned} D(z) &= \mathrm{Det}\left(\frac{z-H}{z-H_0}\right) \\ &= \exp\left(\int_{-\infty}^{\infty} dE \ln(z-E)\left[\rho(E)-\rho_0(E)\right]\right) \end{aligned}\right\} \quad z \text{ complex} \tag{1.1}$$

where $[\rho(E)-\rho_0(E)]$ is the difference in density of states of the two operators:

$$\rho(E)-\rho_0(E) = \mathrm{Tr}\left[\delta(E-H)-\delta(E-H_0)\right]. \tag{1.2}$$

If the Fredholm determinant is known as a function of the complex variable z it may be used to obtain spectral information on H and H_0. For example it is possible to determine the scattering amplitude, the difference in the density of states $[\rho(E)-\rho_0(E)]$ and the difference in free energies associated with H and H_0. In general, if $\rho_0(E)$ is known, any spectral quantity of H that is independent of the eigenvectors of H may also be found.

In the case where both H and H_0 have purely discrete spectra (e.g. for systems defined in a finite region of space), the Fredholm determinant can be written in terms of the eigenvalues E_n (E_{n0}) of H (H_0) as $\prod_n (z-E_n)/(z-E_{n0})$. In this case, if no eigenvalues of H and H_0 coincide, *full* information on the spectra of H and H_0 may be obtained since e.g. eigenvalues of H (H_0) correspond to zeros (poles) of $D(z)$. In the present work we deal with systems defined on an infinite region of space, where the Hamiltonians generally have both continuous and discrete eigenvalues.

The operators H and H_0 may be regarded as matrices of infinite size and from this viewpoint, the Fredholm determinant $D(z)$ is the determinant of the infinite matrix $(z-H)/(z-H_0)$. $D(z)$ therefore appears to be a formidably complicated quantity, however, it was shown in a previous work [1], that the Fredholm determinant associated with a Dirac Hamiltonian appropriate to one spatial dimension could be expressed in terms of the determinant of a finite matrix. The Hamiltonian dealt with in [1] possessed a spatially varying potential and in a "stratified" system where the potential has a piecewise constant variation, the finite matrix, and hence the Fredholm determinant, can be found in closed form.

In the present work we investigate a related problem, namely the Fredholm determinant of a Dirac operator also acting in one-dimension; here without a potential but possessing a spatially varying mass term. We take the mass term in H to have the form[2]

$$\widehat{\Delta}(x) = \Delta_2(x)\widehat{\sigma}^2 + \Delta_3(x)\widehat{\sigma}^3 \tag{1.3}$$

where $\widehat{\sigma}^i$ ($i=1,2,3$) are the Pauli spin matrices (which may be used as a 2×2 representation of the Dirac gamma matrices in one spatial dimension) and x is the spatial coordinate. In contrast to [1], we do not consider the problem defined on a finite interval of the x-axis but on the infinite interval $(-\infty,\infty)$.

A mass term falling into the class of (1.3) has been investigated in the context of fractionally charged solitons in quantum field theory [2], it also appears rather naturally

[1] Throughout this work we use Det and Tr to denote the determinant and trace of infinite matrices (first quantized operators). For the 2×2 matrices appearing in this work, we denote these quantities by det and tr.

[2] As an aid to the identification of the 2×2 matrices in this work, we have labelled them with a caret, $\widehat{\cdot}$. Exceptions to this rule are first quantized operators such as H, which contain an operator content as well as being matrices.

in the condensed matter physics of Fermions[3] coupled e.g. to solitons in linear molecules [3], vortices in superconductors [4] and domain walls in superfluid ^{3}He [5].

All of the examples just listed may have $\widehat{\Delta}(+\infty) \neq \widehat{\Delta}(-\infty)$ and we call any $\widehat{\Delta}(x)$ with this property a "topological mass term" since it splits the x-axis into two separate regions which can only be converted to single region by an rearrangement of $\widehat{\Delta}(x)$ over an infinite distance.

We obtain in this work an expression for the Fredholm determinant when the two operators H and H_0 have mass terms with the *same asymptotic behaviour.* Thus if $\widehat{\Delta}_0(x)$ is the mass term of H_0 then we shall restrict ourselves to the case

$$\begin{aligned} \widehat{\Delta}(+\infty) &= \widehat{\Delta}_0(+\infty), \\ \widehat{\Delta}(-\infty) &= \widehat{\Delta}_0(-\infty). \end{aligned} \tag{1.4}$$

The necessity for this restriction will become apparent in section 3. To keep the calculations general, we will assume that $\widehat{\Delta}(+\infty)$ and $\widehat{\Delta}(-\infty)$ have no relation between them[4].

Central to our calculation of the Fredholm determinant is the determination of the Green's functions

$$G = \frac{1}{z-H}, \qquad G_0 = \frac{1}{z-H_0}, \qquad z \text{ complex}, \tag{1.5}$$

since it follows from Ref.[7] that

$$\begin{aligned} \frac{d}{dz}\ln D(z) &= \frac{d}{dz}\ln \mathrm{Det}\left(\frac{z-H}{z-H_0}\right) = \frac{d}{dz}\mathrm{Tr}\ln\left(\frac{z-H}{z-H_0}\right) \\ &= \mathrm{Tr}\left(\frac{1}{z-H} - \frac{1}{z-H_0}\right) = \mathrm{Tr}\left(G - G_0\right). \end{aligned} \tag{1.6}$$

A suitable representation for the Green's functions will be derived in section 2 of this work. In section 3 the Green's functions will be used to obtain an expression for the Fredholm determinant. The expression obtained depends only on the finite (2×2) matrices characterizing the asymptotic properties of the Green's functions.

Throughout the work the natural constants of c and $\hbar$ are set to unity. Furthermore, the relationship between trace and determinant of a matrix: $\mathrm{tr}\ln \equiv \ln\det$ will be freely used for both finite matrices and those of infinite size (first-quantized operators) .

2. Determination of the Green's function

We regard operators such as H as acting in a space of bras and kets; thus the coordinate representation of the Green's function is $\langle x|G|x'\rangle \equiv \widehat{G}(x,x')$, the 2×2 matrix structure of $\widehat{G}$ being left implicit.

The form we take for H is

$$H = p_x\widehat{\sigma}^1 + \widehat{\Delta}(x) \tag{2.1}$$

where p_x is the momentum operator and $\widehat{\Delta}(x)$ is given by (1.3). The equation of motion for $\widehat{G}(x;x')$ is the coordinate representation of $(z-H)\cdot G = 1$, i.e.

$$\left(z - \left[-i\partial_x\widehat{\sigma}^1 + \widehat{\Delta}(x)\right]\right)\widehat{G}(x;x') = \delta\left(x-x'\right). \tag{2.2}$$

[3]A Dirac-like Hamiltonian often arises as an approximate Hamiltonian of non-relativistic Fermions; the mass term present usually has the interpretation as an order-parameter field.

[4]A problem where there is effectively no relation between the asymptotic limits of $\widehat{\Delta}$ is the A-B phase boundary of superfluid ^{3}He [6] where the A and B phase order parameters lie in different symmetry classes. The principle difference with the A-B phase boundary and the problems considered in the present work lies in the dimension of the matrices used: for the A-B phase boundary matrices of size 4×4 are required.

By multiplying through by $-i\widehat{\sigma}^1$ we obtain

$$\left[\partial_x - \widehat{N}\left(x\right)\right]\widehat{G}(x;x') = -i\widehat{\sigma}^1\delta\left(x-x'\right) \tag{2.3}$$

where

$$\widehat{N}\left(x\right) = i\widehat{\sigma}^1\left[z-\widehat{\Delta}(x)\right]. \tag{2.4}$$

Let us introduce a "transport matrix" $\widehat{M}(x)$ defined by[5]

$$\left[\partial_x - \widehat{N}\left(x\right)\right]\widehat{M}(x) = 0, \qquad \widehat{M}(0) = \widehat{1} \tag{2.5}$$

The transport matrix $\widehat{M}(x)$ carries information about the detailed spatial variation of $\widehat{\Delta}(x)$ as well as information on the asymptotics of this quantity.

Assuming all spatial variation of $\widehat{\Delta}(x)$ takes place in a finite region of the x-axis, the asymptotic behaviour of $\widehat{M}(x)$ can be written as

$$\begin{aligned} \widehat{M}(x\to\infty) &= \exp\left[x\widehat{N}(\infty)\right]\widehat{U} \\ \widehat{M}(x\to-\infty) &= \exp\left[x\widehat{N}(-\infty)\right]\widehat{V} \end{aligned} \tag{2.6}$$

where the matrices $\widehat{U}$ and $\widehat{V}$ have determinants of unity[6]. It is these matrices that encapsulate the cumulative effects of the spatial variation of $\widehat{\Delta}(x)$ and yield a suitably general parametrization of the problem.

In the case where the spatial variation of $\widehat{\Delta}(x)$ takes place over an infinite region of the x-axis, there will be a modification of the asymptotic behaviour given in (2.6); we consider a specific example of this later.

On taking into account the "right" equation of motion $G\cdot(z-H)=1$, in addition to that of the "left," (2.2), we are led to write

$$\begin{aligned} \widehat{G}(x,x') &= \widehat{M}(x)\widehat{A}\widehat{M}^{-1}(x')\left(-i\widehat{\sigma}^1\right), \qquad & x>x' \\ &= -\widehat{M}(x)\widehat{B}\widehat{M}^{-1}(x')\left(-i\widehat{\sigma}^1\right), \qquad & x<x' \end{aligned} \tag{2.7}$$

where the matrices $\widehat{A}$ and $\widehat{B}$ are to be determined. The jump condition, following from (2.3), is

$$\widehat{G}(x;x')\ \Big|_{x=x'_-}^{x=x'_+} = -i\widehat{\sigma}^1 \tag{2.8}$$

and applied to (2.7) yields

$$\widehat{A}+\widehat{B} = \widehat{1}. \tag{2.9}$$

The Green's function $\widehat{G}(x,x')$ is the solution to (2.3) that remains bounded as $|x|$, $|x'|\to\infty$. The eigenvalues of $\widehat{N}(x)$ are written[7] $\pm\lambda\left(x\right)$ where $\mathrm{Re}\left[\lambda\left(x\right)\right]>0$. This results in $\widehat{M}(x)$

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[5] $\widehat{M}(x)$ is termed a transport matrix since it transports "eigenfunctions" $\psi(x)$ of the Dirac equation obeying $\left[-i\partial_x\widehat{\sigma}^1+\widehat{\Delta}(x)\right]\psi(x) = z\psi(x)$ along the x-axis: $\psi(x)=\widehat{M}(x)\psi(0)$.

Note that in contrast to [1], $\widehat{M}(x)$ achieves the value of unity at $x=0$ rather than at the left boundary of the interval, which here is taken to be $x=-\infty$.

[6] Equation (2.5) implies $\det[\widehat{M}(x)] = \det[\widehat{M}(0)]\exp\left(\int_0^x dx'\,\mathrm{tr}[\widehat{N}(x')]\right)$. $\widehat{N}(x)$ has the property $\mathrm{tr}[\widehat{N}(x)]=0$ and, by choice, $\widehat{M}(0)=\widehat{1}$. Thus $\det[\widehat{M}(x)]=1$ and this forces $\widehat{U}$ and $\widehat{V}$ to also have determinants of unity.

[7] With the notation $\|\widehat{\Delta}(x)\|^2 \equiv [\Delta_2(x)]^2+[\Delta_3(x)]^2$, the eigenvalues of $N(x)$ are $\pm\lambda\left(x\right)$ where $\lambda\left(x\right)=\sqrt{\|\widehat{\Delta}(x)\|^2-z^2}$; the square root is cut along the negative real axis and the branch selected has $\mathrm{Re}\left[\lambda\left(x\right)\right]>0$.

having exponentially growing parts for large $|x|$ and it is necessary to eliminate these from the Green's function. We shall use projection operators $\widehat{P}_{\pm}(x)$ with the properties

$$
\begin{aligned}
\widehat{N}(x)\widehat{P}_{\pm}(x) = \pm\lambda(x)\widehat{P}_{\pm}(x) \\
\widehat{P}_{+}(x) + \widehat{P}_{-}(x) = \widehat{1}, \qquad \widehat{P}_{+}(x)\widehat{P}_{-}(x) = 0
\end{aligned}
\tag{2.10}
$$

to achieve this elimination. Taking into account (2.6), we see that a bounded Green's function only results if $\widehat{A}$ and $\widehat{B}$ in (2.7) have the form

$$
\begin{aligned}
\widehat{A} &= \widehat{U}^{-1}\widehat{P}_{-}(\infty)\widehat{A}'\widehat{P}_{-}(-\infty)\widehat{V} \\
\widehat{B} &= \widehat{V}^{-1}\widehat{P}_{+}(-\infty)\widehat{B}'\widehat{P}_{+}(\infty)\widehat{U}
\end{aligned}
\tag{2.11}
$$

with $\widehat{A}'$ and $\widehat{B}'$ additional matrices that are to be determined.

A property of (2.11) is that $\widehat{A}\widehat{B} = 0 = \widehat{B}\widehat{A}$. Combining this with (2.9) implies that $\widehat{A}$ and $\widehat{B}$ are projection operators (neither $\widehat{A}$ nor $\widehat{B}$ can vanish, since the non-vanishing one would be the identity matrix and the Green's function would then become unbounded). Consequently, the eigenvalues of both $\widehat{A}$ and $\widehat{B}$ are 0 and 1 so

$$
\operatorname{tr}\widehat{A} = 1, \qquad \operatorname{tr}\widehat{B} = 1. \tag{2.12}
$$

Applying (2.12) to (2.11) yields

$$
\begin{aligned}
\operatorname{tr}\left[\widehat{U}^{-1}\widehat{P}_{-}(\infty)\widehat{A}'\widehat{P}_{-}(-\infty)\widehat{V}\right] &= 1 \\
\operatorname{tr}\left[\widehat{V}^{-1}\widehat{P}_{+}(-\infty)\widehat{B}'\widehat{P}_{+}(\infty)\widehat{U}\right] &= 1.
\end{aligned}
\tag{2.13}
$$

Without loss of generality, we can choose $\widehat{A}'$ and $\widehat{B}'$ to be a scalar multiple of the identity matrix, or effectively scalars[8], thus we take

$$
A' = \frac{1}{\operatorname{tr}\left[\widehat{U}^{-1}\widehat{P}_{-}(\infty)\widehat{P}_{-}(-\infty)\widehat{V}\right]}, \qquad B' = \frac{1}{\operatorname{tr}\left[\widehat{V}^{-1}\widehat{P}_{+}(-\infty)\widehat{P}_{+}(\infty)\widehat{U}\right]}. \tag{2.14}
$$

It is shown in appendix A that $A' \equiv B'$ and we define

$$
S = \operatorname{tr}\left[\widehat{P}_{+}(\infty)\widehat{U}\widehat{V}^{-1}\widehat{P}_{+}(-\infty)\right] \equiv \operatorname{tr}\left[\widehat{P}_{-}(-\infty)\widehat{V}\widehat{U}^{-1}\widehat{P}_{-}(\infty)\right], \tag{2.15}
$$

then the Green's function is given by

$$
\begin{aligned}
\widehat{G}(x,x') &= \widehat{M}(x)\frac{\widehat{U}^{-1}\widehat{P}_{-}(\infty)\widehat{P}_{-}(-\infty)\widehat{V}}{S}\widehat{M}^{-1}(x')\left(-i\widehat{\sigma}^{1}\right), \qquad x > x' \\
&= -\widehat{M}(x)\frac{\widehat{V}^{-1}\widehat{P}_{+}(-\infty)\widehat{P}_{+}(\infty)\widehat{U}}{S}\widehat{M}^{-1}(x')\left(-i\widehat{\sigma}^{1}\right). \qquad x < x'.
\end{aligned}
\tag{2.16}
$$

In the problem defined on a finite interval, there is a choice of boundary conditions compatible with Hermiticity of the Hamiltonian and consequently a family of possible Green's functions. By contrast, on the infinite line, the Green's function is subject only to the requirement of boundedness and this results in a unique Green's function.

[8] $\widehat{A}'$ and $\widehat{B}'$ can be taken as a scalar multiple of the unit matrix, or effectively, scalars, since e.g. $\widehat{P}_{-}(\infty)\widehat{A}'_{-}\widehat{P}_{-}(-\infty)$ = constant $\times$ $\widehat{P}_{-}(\infty)\widehat{P}_{-}(-\infty)$. Thus the only role of the matrix $\widehat{A}'$ is to fix the value of the constant multiplying $\widehat{P}_{-}(\infty)\widehat{P}_{-}(-\infty)$.

3. Evaluation of the Fredholm Determinant

Equation (1.6) requires the evaluation of

$$\mathrm{Tr}\left(\widehat{G}-\widehat{G}_0\right)\equiv\int\limits_{-\infty}^{\infty}dx\,\mathrm{tr}\left[\widehat{G}(x,x)-\widehat{G}_0(x,x)\right] \tag{3.1}$$

and as an intermediate step we shall determine $\int_{-L_1}^{L_2}dx\,\mathrm{tr}\left[\widehat{G}(x,x)\right]$.

First we note, on taking the trace of (2.3), that $\mathrm{tr}\left[\widehat{G}(x,x')\right]$ is continuous at $x=x'$, thus we can take $\mathrm{tr}\left[\widehat{G}(x,x)\right]$ from e.g. the limit of the $x<x'$ part of the Green's function:

$$\mathrm{tr}\left[\widehat{G}(x,x)\right]=\mathrm{tr}\left[\widehat{M}^{-1}(x)i\widehat{\sigma}^1\widehat{M}(x)\widehat{V}^{-1}\widehat{P}_+(-\infty)\widehat{P}_+(\infty)\widehat{U}\right]/S. \tag{3.2}$$

Equation (3.2) may be written in a useful form by using the following result, which follows from differentiating (2.5) with respect to[9] z,

$$\widehat{M}^{-1}(x)i\widehat{\sigma}^1\widehat{M}(x)=\partial_x\left(\widehat{M}^{-1}(x)\frac{d\widehat{M}(x)}{dz}\right). \tag{3.3}$$

When used in (3.2), (3.3) yields

$$\mathrm{tr}\left[\widehat{G}(x,x)\right]=\partial_x\,\mathrm{tr}\left[\widehat{M}^{-1}(x)\frac{d\widehat{M}(x)}{dz}\frac{\widehat{V}^{-1}\widehat{P}_+(-\infty)\widehat{P}_+(\infty)\widehat{U}}{S}\right] \tag{3.4}$$

and this last equation can be integrated[10]:

$$\begin{aligned}\int_{-L_1}^{L_2}dx\,\mathrm{tr}\left[\widehat{G}(x,x)\right] \quad &= \quad \mathrm{tr}\left[\widehat{M}^{-1}(L_2)\frac{d\widehat{M}(L_2)}{dz}\frac{\widehat{V}^{-1}\widehat{P}_+(-\infty)\widehat{P}_+(\infty)\widehat{U}}{S}\right]\\ &\quad +\mathrm{tr}\left[\frac{d\widehat{M}^{-1}(-L_1)}{dz}\widehat{M}(-L_1)\frac{\widehat{V}^{-1}\widehat{P}_+(-\infty)\widehat{P}_+(\infty)\widehat{U}}{S}\right].\end{aligned} \tag{3.5}$$

We assume that L_1 and L_2 are sufficiently large that $\widehat{M}(-L_1)$ and $\widehat{M}(L_2)$ have achieved

[9] Differentiating (2.5) with respect to z yields $\left[\partial_x-\widehat{N}(x)\right]d\widehat{M}(x)/dz=i\sigma^1\widehat{M}(x)$. The left side of this equation can be written as $\left[\partial_x-\widehat{N}(x)\right]\left[\widehat{M}(x)\widehat{M}^{-1}(x)d\widehat{M}(x)/dz\right]=\widehat{M}(x)\partial_x\left[\widehat{M}^{-1}(x)d\widehat{M}(x)/dz\right]$ and from this (3.3) follows.

Eq. (3.3) applies for all values of x and by taking a limiting procedure, it may be explicitly verified that (3.3) applies for the particular case $x=0$.

[10] For the second term in (3.5) we have found it convenient to replace $\widehat{M}^{-1}d\widehat{M}/dz$ by $-\left(d\widehat{M}^{-1}/dz\right)\widehat{M}$.

their asymptotic forms (2.6), then e.g.

$$
\begin{aligned}
&\operatorname{tr}\left[\widehat{M}^{-1}(L_2)\frac{d\widehat{M}(L_2)}{dz}\widehat{V}^{-1}\widehat{P}_+(-\infty)\widehat{P}_+(\infty)\widehat{U}\right] \\
&= \operatorname{tr}\left[\widehat{U}^{-1}e^{-L_2\widehat{N}(\infty)}\frac{d}{dz}\left(e^{L_2\widehat{N}(\infty)}\widehat{U}\right)\widehat{V}^{-1}\widehat{P}_+(-\infty)\widehat{P}_+(\infty)\widehat{U}\right] \\
&= \operatorname{tr}\left[e^{-L_2\lambda(\infty)}\frac{d}{dz}\left[\left(e^{L_2\lambda(\infty)}\widehat{P}_+(\infty)+e^{-L_2\lambda(\infty)}\widehat{P}_-(\infty)\right)\widehat{U}\right]\widehat{V}^{-1}\widehat{P}_+(-\infty)\widehat{P}_+(\infty)\right] \\
&= SL_2\frac{d\lambda(\infty)}{dz}+\operatorname{tr}\left[\frac{d}{dz}\left[\widehat{P}_+(\infty)\widehat{U}\right]\widehat{V}^{-1}\widehat{P}_+(-\infty)\widehat{P}_+(\infty)\right]+O(e^{-2L_2\lambda(\infty)}) \\
&= SL_2\frac{d\lambda(\infty)}{dz}+\operatorname{tr}\left[\widehat{P}_+(-\infty)\widehat{P}_+(\infty)\frac{d}{dz}\left[\widehat{P}_+(\infty)\widehat{U}\right]\widehat{V}^{-1}\widehat{P}_+(-\infty)\right]+O(e^{-2L_2\lambda(\infty)})
\end{aligned}
\tag{3.6}
$$

(in the last equation we have used $\widehat{P}_+(-\infty)=\left[\widehat{P}_+(-\infty)\right]^2$ and cyclic invariance of the trace). Henceforth, we shall neglect exponentially small terms which will ultimately vanish when the limits $L_{1,2}\to\infty$ are taken.

Proceeding as above with the second term in (3.5) we find we can write

$$
\begin{aligned}
&\int_{-L_1}^{L_2} dx\,\operatorname{tr}\left[\widehat{G}(x,x)\right] \\
&= \operatorname{tr}\left[\frac{\widehat{P}_+(-\infty)\widehat{P}_+(\infty)}{S}\frac{d}{dz}\left[\widehat{P}_+(\infty)\widehat{U}\widehat{V}^{-1}\widehat{P}_+(-\infty)\right]\right]+\frac{d}{dz}\left[L_2\lambda(\infty)+L_1\lambda(-\infty)\right] \\
&= \operatorname{tr}\left[\frac{1}{S}\frac{d}{dz}\left[\widehat{P}_+(\infty)\widehat{U}\widehat{V}^{-1}\widehat{P}_+(-\infty)\right]\right]+\frac{d}{dz}\left[L_2\lambda(\infty)+L_1\lambda(-\infty)\right] \\
&\quad -\operatorname{tr}\left[\frac{\widehat{P}_+(\infty)\widehat{U}\widehat{V}^{-1}\widehat{P}_+(-\infty)}{S}\frac{d}{dz}\left[\widehat{P}_+(-\infty)\widehat{P}_+(\infty)\right]\right].
\end{aligned}
\tag{3.7}
$$

The last term in this expression is independent of $\widehat{U}$ and $\widehat{V}$ since $\widehat{P}_+(\infty)\widehat{U}\widehat{V}^{-1}\widehat{P}_+(-\infty)/S=\text{constant}\times\widehat{P}_+(\infty)\widehat{P}_+(-\infty)$ and taking the trace of each side of this equation indicates that the constant has the value $1/\operatorname{tr}\left[\widehat{P}_+(\infty)P_+(-\infty)\right]$. Equation (3.7) therefore yields

$$
\begin{aligned}
\int_{-L_1}^{L_2} dx\,\operatorname{tr}\left[\widehat{G}(x,x)\right] &= \frac{d}{dz}\ln\left[\operatorname{tr}\left(\widehat{P}_+(\infty)\widehat{U}\widehat{V}^{-1}\widehat{P}_+(-\infty)\right)\right] \\
&\quad + \text{terms independent of } \widehat{U} \text{ and } \widehat{V}.
\end{aligned}
\tag{3.8}
$$

The terms independent of $\widehat{U}$ and $\widehat{V}$ depend on the projection operators $P_+(\pm\infty)$ and these are determined solely by the asymptotic limits of $\widehat{\Delta}(x)$. These terms are, in general, only guaranteed to cancel with corresponding terms in $\int_{-L_1}^{L_2} dx\,\operatorname{tr}\left[\widehat{G}_0(x,x)\right]$ providing the asymptotic limits of $\widehat{\Delta}_0(x)$ are identical to those of $\widehat{\Delta}(x)$. This is the origin of the restriction $\widehat{\Delta}(\pm\infty)=\widehat{\Delta}_0(\pm\infty)$ that we imposed on the masses.

Proceeding, we find in the limit of infinite L_1 and L_2:

$$\int\limits_{-\infty}^{\infty} dx\,\mathrm{tr}\left[\widehat{G}(x,x)-\widehat{G}_0(x,x)\right]=\frac{d}{dz}\ln\left[\frac{\mathrm{tr}\left(\widehat{P}_+(\infty)\widehat{U}\widehat{V}^{-1}\widehat{P}_+(-\infty)\right)}{\mathrm{tr}\left(\widehat{P}_+(\infty)\widehat{U}_0\widehat{V}_0^{-1}\widehat{P}_+(-\infty)\right)}\right] \tag{3.9}$$

and using (1.6) we have

$$\frac{d}{dz}\ln\mathrm{Det}\left(\frac{z-H}{z-H_0}\right)=\frac{d}{dz}\ln\left[\frac{\mathrm{tr}\left(\widehat{P}_+(\infty)\widehat{U}\widehat{V}^{-1}\widehat{P}_+(-\infty)\right)}{\mathrm{tr}\left(\widehat{P}_+(\infty)\widehat{U}_0\widehat{V}_0^{-1}\widehat{P}_+(-\infty)\right)}\right]. \tag{3.10}$$

For non-singular mass terms in H and H_0, it may be proved that in the limit of "high energies," where $|z|\to\infty$ along any ray through the origin, not coinciding with the real axis, $\dfrac{\mathrm{tr}\left(\widehat{P}_+(\infty)\widehat{U}\widehat{V}^{-1}\widehat{P}_+(-\infty)\right)}{\mathrm{tr}\left(\widehat{P}_+(\infty)\widehat{U}_0\widehat{V}_0^{-1}\widehat{P}_+(-\infty)\right)}$ tends to unity. $\mathrm{Det}\left(\dfrac{z-H}{z-H_0}\right)$ also shares this property and this allows us to integrate (3.10) with the result

$$\mathrm{Det}\left(\frac{z-H}{z-H_0}\right)=\frac{\mathrm{tr}\left(\widehat{P}_+(\infty)\widehat{U}\widehat{V}^{-1}\widehat{P}_+(-\infty)\right)}{\mathrm{tr}\left(\widehat{P}_+(\infty)\widehat{U}_0\widehat{V}_0^{-1}\widehat{P}_+(-\infty)\right)}. \tag{3.11}$$

This is our principal conclusion. We see that the considerable amount of information contained in the Fredholm determinant is also contained in the 2×2 matrices characterizing the asymptotic spatial behaviour of $\widehat{G}(x,x')$.

The above equation gives the Fredholm determinant when the spatial variation of $\widehat{\Delta}(x)$ is contained in a finite region of space. In appendix B, we consider $\widehat{\Delta}(x)=\widehat{\sigma}^2\tanh(x)$ as a simple, exactly soluble example where the mass term has a variation over all space.

It is interesting to note that there is a formula very closely related (3.11) for one-dimensional *Schrödinger* Hamiltonians and in appendix C we outline the derivation of this result.

4. Discussion

In this work we have considered a pair of Dirac operators and derived a relation between their Fredholm determinant and the matrices characterizing the asymptotic spatial properties of the associated Green's functions. As in [1] an effective reduction in the dimensionality is exhibited: the determinant of an infinite matrix is related to a simple function of finite matrices.

Of the 2×2 matrices on which the final result (3.11) depends, the projection operators $\widehat{P}_\pm(\infty)$ are simple constructs of $\widehat{\Delta}(\pm\infty)$ and require no calculation. The matrices carrying non-trivial information are $\widehat{U}$ and $\widehat{V}$ and these can only be found by "integrating" the transport equation (2.5) out to distances beyond the range of variation of $\widehat{\Delta}(x)$. As pointed out in [1], the transport matrix $\widehat{M}(x)$ may be calculated in closed form in a stratified system where $\widehat{\Delta}(x)$ has a piecewise constant variation and this may be adequate for some of the applications of this work.

A quantity which has been studied in connection with Dirac operators is the fractional Fermionic charge; a quantity proportional to the spectral asymmetry $\sum_n \mathrm{sgn}(E_n)$ (see e.g. Ref.[8] and references therein). The spectral asymmetry may be shown to be a topological invariant determined only by the asymptotic limits of $\widehat{\Delta}(x)$. Thus the restriction on the mass terms, which arose naturally in the course of the calculation, $\widehat{\Delta}(\pm\infty)=\widehat{\Delta}_0(\pm\infty)$, implies that the mass terms in both H and H_0 lie in the same topological sector and correspond to the same spectral asymmetry (or fractional charge). The Fredholm determinant investigated in this paper therefore summarizes the effects of local differences in the mass terms of the two Hamiltonians.

Let us end this work by briefly discussing a use of the results presented here. In the determination of the profile $\widehat{\Delta}(x)$ of a "soliton" coupled to Fermions, it is necessary to minimise the free energy of the system. A quantity that appears in the free energy (see e.g. [4]) is $\ln D(i\omega_m) \equiv \ln \mathrm{Det}\left(\frac{i\omega_m - H}{i\omega_m - H_0}\right)$ where ω_m are Matsubara frequencies. For many cases of interest $\widehat{\Delta}(-\infty) \neq \widehat{\Delta}(+\infty)$ and the results of this work may be directly used to express $\ln D(i\omega_m)$ in terms of the 2×2 matrices underlying the problem. In comparison with the first quantized operators (with their infinite number of matrix elements and eigenvalues), the 2×2 matrices may be far easier to use and also more readily approximated.

Appendices

A. Proof of $\mathrm{tr}\left[\widehat{U}^{-1}\widehat{P}_{-}(\infty)\widehat{P}_{-}(-\infty)\widehat{V}\right] = \mathrm{tr}\left[\widehat{V}^{-1}\widehat{P}_{+}(-\infty)\widehat{P}_{+}(\infty)\widehat{U}\right]$

In this appendix, we provide a proof of the above relation; a result used in section 2 of this work.

We shall give the proof in two steps:

(i) we show that $\mathrm{tr}\left[\widehat{U}^{-1}\widehat{P}\widehat{V}\right] = \mathrm{tr}\left[\widehat{V}^{-1}\widehat{Q}\widehat{U}\right]$ where $\widehat{P}$ and $\widehat{Q}$ are 2×2 projection operators obeying $\widehat{P}+\widehat{Q}=\widehat{1}$, $\widehat{P}\widehat{Q}=0$.

(ii) we show that we can make the identifications $\widehat{P}_{-}(\infty)\widehat{P}_{-}(-\infty)=\alpha\widehat{P}$, $\widehat{P}_{+}(-\infty)\widehat{P}_{+}(\infty)=\alpha\widehat{Q}$ with α a scalar.

The combination of (i) and (ii) yields the required result.

A.1. Proof that $\mathrm{tr}\left[\widehat{U}^{-1}\widehat{P}\widehat{V}\right] = \mathrm{tr}\left[\widehat{V}^{-1}\widehat{Q}\widehat{U}\right]$

Let us define

$$\widehat{T} = \widehat{V}\widehat{U}^{-1} \tag{A.1}$$

and because $\widehat{U}$ and $\widehat{V}$ have determinants of unity,

$$\det\widehat{T} = 1. \tag{A.2}$$

We thus have to show that

$$\mathrm{tr}\left[\widehat{P}\widehat{T}\right] = \mathrm{tr}\left[\widehat{Q}\widehat{T}^{-1}\right]. \tag{A.3}$$

Consider

$$\begin{aligned}\mathrm{tr}\left[\widehat{P}\widehat{T}\right] - \mathrm{tr}\left[\widehat{Q}\widehat{T}^{-1}\right] &= \mathrm{tr}\left[\widehat{P}\widehat{T}\right] - \mathrm{tr}\left[\left(\widehat{1}-\widehat{P}\right)\widehat{T}^{-1}\right]\\ &= \mathrm{tr}\left[\widehat{P}\left(\widehat{T}+\widehat{T}^{-1}\right)\right] - \mathrm{tr}\left[\widehat{T}^{-1}\right].\end{aligned} \tag{A.4}$$

Because of (A.2), we have[11]

$$\widehat{T}+\widehat{T}^{-1} = \mathrm{tr}\left[\widehat{T}^{-1}\right]\cdot\widehat{1} \tag{A.5}$$

and using

$$\mathrm{tr}\left[\widehat{P}\right] = 1, \tag{A.6}$$

we have that the right side of (A.4) identically zero and (A.3) is established.

A.2. Identifying $\widehat{P}_{-}(\infty)\widehat{P}_{-}(-\infty)$ and $\widehat{P}_{+}(-\infty)\widehat{P}_{+}(\infty)$ with projection operators

To prove that $\widehat{P}_{-}(\infty)\widehat{P}_{-}(-\infty)$ and $\widehat{P}_{+}(-\infty)\widehat{P}_{+}(\infty)$ are, up to a common factor, complete and orthogonal projection operators we need to establish orthogonality and completeness.

Let us write

$$\widehat{p} = \widehat{P}_{-}(\infty)\widehat{P}_{-}(-\infty), \quad \widehat{q} = \widehat{P}_{+}(-\infty)\widehat{P}_{+}(\infty) \tag{A.7}$$

then orthogonality entails

$$\widehat{p}\widehat{q} = 0 = \widehat{q}\widehat{p} \tag{A.8}$$

and on inspection this is seen to hold.

[11] The result in (A.5) may be simply proved by applying the Cayley–Hamilton theorem to $\widehat{T}^{-1}$. This states that $\widehat{T}^{-1}$ obeys $\left(\widehat{T}^{-1}\right)^2 - \widehat{T}^{-1}\,\mathrm{tr}\left(\widehat{T}^{-1}\right) + \det\left(\widehat{T}^{-1}\right)\widehat{1} = 0$. Mutliplying this equation by $\widehat{T}$, using (A.2) and rearranging immediately yields (A.5).

To prove completeness, we need to show

$$\widehat{p} + \widehat{q} = \text{scalar} \times \widehat{1}. \tag{A.9}$$

A simple way to demonstrate this is to write

$$\widehat{P}_{\pm}(\pm\infty) = \frac{1}{2}\left[\widehat{1} \pm \overrightarrow{\omega}(\pm\infty) \cdot \overrightarrow{\sigma}\right], \quad \text{all combinations of signs,} \tag{A.10}$$

where

$$\overrightarrow{\omega} \cdot \overrightarrow{\sigma} = \omega_1\sigma^1 + \omega_2\sigma^2 + \omega_3\sigma^3. \tag{A.11}$$

Using (A.10) and (A.7) yields, from the properties of the Pauli matrices,

$$\widehat{p} + \widehat{q} = \frac{1}{2}\left[1 + \overrightarrow{\omega}(-\infty) \cdot \overrightarrow{\omega}(\infty)\right] \times \widehat{1}. \tag{A.12}$$

Thus with (A.8) and (A.12) we have established that $\widehat{P}_{-}(\infty)\widehat{P}_{-}(-\infty)$ and $\widehat{P}_{+}(-\infty)\widehat{P}_{+}(\infty)$ are, up to a common scalar factor, a complete and orthogonal set of projection operators.

With the above results we have proved the equation in the title of this appendix.

Let us state an interesting extension of the results of this appendix which underlies the construction of the Green's function in section 2. Let $\widehat{P}$ and $\widehat{Q}$ be 2×2 matrix projection operators with $\widehat{P} + \widehat{Q} = \widehat{1}$, $\widehat{P}\widehat{Q} = 0$ and let $\widehat{U}$ and $\widehat{V}$ be 2×2 matrices of determinant unity. If $\widehat{P}' \equiv \widehat{U}^{-1}\widehat{P}\widehat{V}/\operatorname{tr}\left(\widehat{U}^{-1}P\widehat{V}\right)$, $\widehat{Q}' \equiv \widehat{V}^{-1}\widehat{Q}\widehat{U}/\operatorname{tr}\left(\widehat{V}^{-1}\widehat{Q}\widehat{U}\right)$, then $\widehat{P}'$ and $\widehat{Q}'$ are also projection operators obeying $\widehat{P}' + \widehat{Q}' = \widehat{1}$, $\widehat{P}'\widehat{Q}' = 0$.

B. Calculations for the mass term $\widehat{\Delta}(x) = \widehat{\sigma}^2 \tanh(x)$

In this appendix, we calculate the Fredholm determinant for the mass term

$$\widehat{\Delta}(x) = \widehat{\sigma}^2 \tanh(x). \tag{B.1}$$

Up to a unitary transformation and a rescaling of lengths and energies, the above choice for $\widehat{\Delta}(x)$ corresponds to the topological soliton or kink that exists in the dimerization of the linear molecule trans-polyacetylene [3].

Associated with (B.1) is the matrix

$$\widehat{N}(x) \equiv i\widehat{\sigma}^1\left[z - \widehat{\Delta}(x)\right] = i\widehat{\sigma}^1 z + \widehat{\sigma}^3 \tanh(x) \tag{B.2}$$

with the asymptotic values

$$\widehat{N}(\pm\infty) = i\widehat{\sigma}^1 z \pm \widehat{\sigma}^3. \tag{B.3}$$

$\widehat{N}(\pm\infty)$ both have eigenevalues $\pm\lambda$ where

$$\lambda = \sqrt{1 - z^2}, \tag{B.4}$$

and the same definition for the square root as given in footnote 7 is used.

The transport matrix $\widehat{M}(x)$, (2.5), is found to be

$$\widehat{M}(x) = \begin{pmatrix} \cosh(\lambda x) & \dfrac{iz}{\lambda}\sinh(\lambda x) \\ \dfrac{\lambda\sinh(\lambda x) - \tanh(x)\cosh(\lambda x)}{iz} & \dfrac{\lambda\cosh(\lambda x) - \tanh(x)\sinh(\lambda x)}{\lambda} \end{pmatrix}. \tag{B.5}$$

Let us consider the behaviour of $\widehat{M}(x)$ for large positive x. Using the abbreviations

$$c = \cosh(\lambda x), \quad s = \sinh(\lambda x), \quad t = \tanh(x) \tag{B.6}$$

we can write, with no approximation,

$$\widehat{M}(x) = \widehat{M}_\infty(x) + (1-t)\begin{pmatrix} 0 & 0 \\ \dfrac{c}{iz} & \dfrac{s}{\lambda} \end{pmatrix}, \quad \widehat{M}_\infty(x) = \begin{pmatrix} c & \dfrac{izs}{\lambda} \\ \dfrac{\lambda s - c}{iz} & \dfrac{\lambda c - s}{\lambda} \end{pmatrix}. \tag{B.7}$$

$\widehat{M}_\infty(x)$ is the matrix obtained by setting $\tanh(x) = +1$ in $\widehat{M}(x)$. It is evident from (B.7) that $\widehat{M}(x)$ does not behave as $e^{x\widehat{N}(\infty)}\widehat{U}$ for large positive x: $\widehat{M}_\infty(x)$ can be shown to behave in this way but $(1-t)\begin{pmatrix} 0 & 0 \\ \dfrac{c}{iz} & \dfrac{s}{\lambda} \end{pmatrix}$ contains terms involving $e^{-2nx\pm\lambda x}$ ($n = 1, 2, 3...$) and for some values of z, $e^{-2nx+\lambda x}$ may grow exponentially with x. This appears to be a problem, since the requirement of boundedness of the Green's function forced us to project away the exponentially growing part of $\widehat{M}(x)$, and now we apparently have more than one part that grows exponentially with x. That this is not a problem becomes clear if we note that (B.7) can be written as

$$\widehat{M}(x) = \left[\widehat{1} + \frac{(1-t)}{iz}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]\widehat{M}_\infty(x) \tag{B.8}$$

and therefore projecting away the growing part of $\widehat{M}_\infty(x)$ automatically eliminates *all* growing parts of $\widehat{M}(x)$.

It is thus natural to generalize (2.6) by defining a matrix $\widehat{U}$ via $\widehat{M}_\infty(x)$:

$$\widehat{M}_\infty(x) = \exp\left[x\widehat{N}(\infty)\right]\widehat{U} \tag{B.9}$$

(in the present example, this form holds *for all* x and not just large positive x). A related matrix, $\widehat{V}$, is defined in terms of $\widehat{M}_{-\infty}(x)$ which is obtained from $\widehat{M}(x)$ by setting $\tanh(x) = -1$:

$$\widehat{M}_{-\infty}(x) \equiv \begin{pmatrix} c & \dfrac{izs}{\lambda} \\ \dfrac{\lambda s + c}{iz} & \dfrac{\lambda c + s}{\lambda} \end{pmatrix} = \exp\left[x\widehat{N}(-\infty)\right]\widehat{V} \tag{B.10}$$

and we find that

$$\widehat{U} = \begin{pmatrix} 1 & 0 \\ i/z & 1 \end{pmatrix}, \qquad \widehat{V} = \begin{pmatrix} 1 & 0 \\ -i/z & 1 \end{pmatrix}. \tag{B.11}$$

To calculate the Fredholm determinant, we write

$$\begin{aligned} \widehat{M}(x) &= \widehat{O}_\infty(x)\exp\left[x\widehat{N}(\infty)\right]\widehat{U} \\ \widehat{M}(x) &= \widehat{O}_{-\infty}(x)\exp\left[x\widehat{N}(-\infty)\right]\widehat{V} \end{aligned} \tag{B.12}$$

where $\widehat{O}_\infty(x)$ may be identified from (B.8) and $\widehat{O}_{-\infty}(x)$ follows from the analogous equation involving $\widehat{M}_{-\infty}(x)$.

Note that

$$\lim_{x\to\pm\infty}\widehat{O}_{\pm\infty}(x) = \widehat{1}, \qquad \det\left[\widehat{O}_{\pm\infty}(x)\right] = 1 \tag{B.13}$$

and these properties allow the method of section 3 to be applied, and lead to a result of identical form to that of (3.11) (but with $\widehat{U}$ and $\widehat{V}$ defined by (B.9, B.10)):

$$\mathrm{Det}\left(\frac{z-H}{z-H_0}\right) = \frac{\mathrm{tr}\left(\widehat{P}_+(\infty)\widehat{U}\widehat{V}^{-1}\widehat{P}_+(-\infty)\right)}{\mathrm{tr}\left(\widehat{P}_+(\infty)\widehat{U}_0\widehat{V}_0^{-1}\widehat{P}_+(-\infty)\right)}. \tag{B.14}$$

Let us evaluate the above formula when the mass term appearing in H_0, $\widehat{\Delta}_0(x)$, is given by

$$\widehat{\Delta}_0(x) = \widehat{\sigma}^2\,\mathrm{sgn}(x). \tag{B.15}$$

In this case

$$\widehat{U}_0 = \widehat{1}, \qquad \widehat{V}_0 = \widehat{1} \tag{B.16}$$

and we obtain

$$\mathrm{Det}\left(\frac{z-H}{z-H_0}\right) = \frac{\sqrt{1-z^2}}{\sqrt{1-z^2}+1}. \tag{B.17}$$

More generally, if other choices of the mass term lead to a transport matrix with the asymptotic forms

$$\begin{aligned}\widehat{M}(x\to\infty) &= \widehat{O}_\infty(x)\exp\left[x\widehat{N}(\infty)\right]\widehat{U} \\ \widehat{M}(x\to-\infty) &= \widehat{O}_{-\infty}(x)\exp\left[x\widehat{N}(-\infty)\right]\widehat{V}\end{aligned} \tag{B.18}$$

and $\widehat{O}_{\pm\infty}(x)$ have the properties of (B.13), then (B.14) will again yield the relationship between the Fredholm determinant and the 2×2 matrices characterizing the problem.

C. Schrödinger Hamiltonian

In this appendix, we provide a derivation of an expression for the Fredholm determinant for a one dimensional Schrödinger operator with a spatially varying potential $v(x)$, showing how closely the results resemble those of the Dirac operator.

To avoid a proliferation of symbols, we shall use the same symbols in this appendix as in the body of the paper with the understanding that they are defined for the Schrödinger problem rather than the Dirac one.

The Green's function for a Schrödinger Hamiltonian, $G(x,x')$, is taken to obey

$$\left[z-\left(-\partial_x^2+v(x)\right)\right]G(x,x') = \delta(x-x'), \tag{C.1}$$

and on introducing

$$F(x)\equiv F(x,x') = \begin{pmatrix} G(x,x') \\ \partial_x G(x,x') \end{pmatrix} \tag{C.2}$$

we can write (C.1) as

$$\left[\partial_x - \widehat{N}(x)\right]F(x) = \delta(x-x')\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{C.3}$$

where[12]

$$\widehat{N}(x) = \begin{pmatrix} 0 & 1 \\ v(x)-z & 0 \end{pmatrix}. \tag{C.4}$$

[12]The eigenvalues of the Schrödinger $\widehat{N}(x)$ matrix are $\pm\sqrt{v(x)-z}$; i.e. different from those of the Dirac problem.

Next we define a 2×2 transfer matrix $M(x)$ obeying (2.5) with $\widehat{N}(x)$ now given by (C.4).

Assuming all variation of the potential takes place in a finite region of the x-axis, asymptotic behaviour of the form of (2.6) applies. It then follows, by very similar considerations to the Dirac case, that

$$\begin{aligned} G(x,x') &= \begin{pmatrix} 1, & 0 \end{pmatrix} \widehat{M}(x)\frac{\widehat{U}^{-1}\widehat{P}_-(\infty)\widehat{P}_-(-\infty)\widehat{V}}{S}\widehat{M}^{-1}(x')\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad x > x' \\ &= -\begin{pmatrix} 1, & 0 \end{pmatrix} \widehat{M}(x)\frac{\widehat{V}^{-1}\widehat{P}_+(-\infty)\widehat{P}_+(\infty)\widehat{U}}{S}\widehat{M}^{-1}(x')\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x < x'. \end{aligned} \tag{C.5}$$

Note that for any 2×2 matrix, A, we have $\begin{pmatrix} 1, & 0 \end{pmatrix} A \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \mathrm{tr}\left[A\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]$ and furthermore, analogous to (3.3), we have

$$\widehat{M}^{-1}(x)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\widehat{M}(x) = -\partial_x\left(\widehat{M}^{-1}(x)\frac{d\widehat{M}(x)}{dz}\right). \tag{C.6}$$

It follows that

$$G(x,x) = \partial_x\,\mathrm{tr}\left[\widehat{M}^{-1}(x)\frac{d\widehat{M}(x)}{dz}\frac{\widehat{V}^{-1}\widehat{P}_+(-\infty)\widehat{P}_+(\infty)\widehat{U}}{S}\right] \tag{C.7}$$

and consequently

$$\mathrm{Det}\left(\frac{z-H}{z-H_0}\right) = \frac{\mathrm{tr}\left(\widehat{P}_+(\infty)\widehat{U}\widehat{V}^{-1}\widehat{P}_+(-\infty)\right)}{\mathrm{tr}\left(\widehat{P}_+(\infty)\widehat{U}_0\widehat{V}_0^{-1}\widehat{P}_+(-\infty)\right)} \tag{C.8}$$

i.e. identical in *form* to the Dirac result, (3.11).

As an illustrative example, we consider the potential

$$v(x) = v_0\delta(x-a), \qquad a > 0 \tag{C.9}$$

($\widehat{M}(x)$ has its value specified at $x=0$ and it would introduce inessential distractions to place the delta function at the same point so we have located it elsewhere). Taking the delta function as the limit of a local function of x, we find

$$\widehat{U} = \widehat{1} + e^{-a\widehat{N}(\infty)}\begin{pmatrix} 0 & 0 \\ v_0 & 0 \end{pmatrix}e^{a\widehat{N}(\infty)}, \qquad \widehat{V} = \widehat{1}. \tag{C.10}$$

Taking H_0 to correspond to a Hamiltonian with zero potential, we find, from (C.8) and

$$\widehat{P}_+(\pm\infty) = \frac{1}{2}\left(\widehat{1} + \frac{\widehat{N}(\pm\infty)}{\sqrt{-z}}\right), \qquad \widehat{N}(\pm\infty) = \begin{pmatrix} 0 & 1 \\ -z & 0 \end{pmatrix}, \tag{C.11}$$

that

$$\mathrm{Det}\left(\frac{z-H}{z-H_0}\right) = 1 + \frac{v_0}{\sqrt{-z}}. \tag{C.12}$$

This result for the Fredholm determinant coincides with the result found by other methods.

Acknowledgments

It is a pleasure to thank G Barton, M K Harder and G Williams for stimulating and helpful discussions.

This work was supported by the Engineering and Physical Sciences Research Council (U.K.).

References

[1] D Waxman, *Ann. Phys.* **231** (1994) 256.

[2] R Jackiw and G Rebbi, *Phys. Rev.* **D18** (1976) 3398.

[3] H Takayama, Y R Lin-Liu and K. Maki, *Phys. Rev.* **B21** (1980) 2388

[4] D Waxman and G Williams, *Ann. Phys.* **220** (1992) 274.

[5] T L Ho, J R Fulco, J R Schrieffer and F Wilczek, *Phys. Rev. Lett.* **52** (1984) 1524

[6] N Schopohl and D Waxman, *Phys. Rev. Lett.* **63** (1989) 1696.

[7] B Sakita, "Quantum Theory of Many-variable systems and Fields," Sect. IV, World Scientific, Singapore, 1985.

[8] M Stone, *Phys. Rev.* **B31**, (1984) 6112.