

The Fredholm Determinant for a Dirac Operator

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The Fredholm determinant for a Dirac operator appropriate to a particle moving in one spatial dimension is investigated. The operator is written as $H = p_x \sigma_1 + m \sigma_3 + V(x)$, where p_x , m , and $V(x)$ are, respectively, the momentum, mass, and potential energy of the particle and the Pauli spin matrices, σ_i , constitute a representation of the Dirac matrices. With $H_0 = p_x \sigma_1 + m \sigma_3$ and z a complex number, the Fredholm determinant is denoted by $\text{Det}[(z - H)/(z - H_0)]$. Let $M(x)$ be the 2×2 matrix that transfers a spinor solution, $\psi(x)$, of the Dirac equation $H\psi(x) = z\psi(x)$ from $-L$ to x : $\psi(x) = M(x)\psi(-L)$ and let $M_0(x)$ be the corresponding matrix for H_0 . Then it is shown, for eigenfunctions obeying the periodic boundary condition $\psi(L) = \psi(-L)$, that $\text{Det}[(z - H)/(z - H_0)]$ equals the determinant of the 2×2 matrix $[1 - M(L)]/[1 - M_0(L)]$. The calculation of an infinite determinant is thus reduced to the calculation of a 2×2 determinant and for piecewise constant potentials an expression for $\text{Det}[(z - H)/(z - H_0)]$ may be derived in closed form. The relation between the Fredholm determinant and the finite determinant was conjectured in an earlier work by D. Waxman and K. D. Ivanova-Moser, *Ann. Phys.* **226** (1993), 271. © 1994 Academic Press, Inc.

1. INTRODUCTION

The Dirac equation involves a first quantised Hamiltonian in the form of a matrix whose elements are linear in the momentum operator. It was introduced by Dirac to incorporate special relativity into quantum mechanics and is applicable to fermions such as electrons. The motivation for the present work did not arise from a relativistic context. It arose from work on Fermi systems which are superfluid at low temperatures [1] and whose particles are non-relativistic. A number of these superfluid Fermi systems are well described by a Dirac Hamiltonian appropriate to a particle moving in one spatial dimension.¹

¹ Non-relativistic systems including the linear molecule, polyacetylene, and some Fermi superfluids are approximately describable in terms of a Dirac Hamiltonian. For the Fermi superfluids this approximate description has the following origin. The interacting fermions of such systems (including ³He, a dilute solution of ³He in ⁴He and also electrons in metals) are understood to undergo a low temperature phase transition into a superfluid state. The broken symmetry of such a state is signalled by the appearance of an order parameter which results in the first quantised Hamiltonian for the system, the Bogoliubov Hamiltonian, possessing a non-trivial matrix structure. Since the particles in question are non-relativistic, the Bogoliubov Hamiltonian has a quadratic dependence on momentum. Often, however, this may be well approximated by a linear momentum dependence (see, e.g., Appendix A of Ref. [1] or Ref. [2]) and once this linearization is made the resulting Hamiltonian is a matrix differential operator, identical in form to a Dirac Hamiltonian with the role of the mass played by the order parameter.

Given the first quantised Hamiltonian for a system, some physically interesting quantities require information on the spectrum, but not the eigenvectors of this operator. For example, in the context of relativistic quantum field theory and also the theory of the linear molecule, polyacetylene, the fractional fermionic charge of a topological soliton is directly related to the spectral asymmetry of the Hamiltonian [3]. In a condensed state context, the free energy and global density of states of a system described by the Hamiltonian are determined by its eigenvalues.

An object that encapsulates full information on the spectrum of an operator is the functional or Fredholm determinant associated with the problem.²

If H denotes the operator of interest, with eigenvalues E_n , and a suitable reference operator H_0 has eigenvalues E_{n0} , then the Fredholm determinant associated with the problem is denoted by $D(z)$ and³

$$D(z) = \text{Det} \left(\frac{z - H}{z - H_0} \right) \equiv \prod_n \left(\frac{z - E_n}{z - E_{n0}} \right), \quad z \text{ complex.} \quad (1.1)$$

We note that in Ref. [1], the density of states was calculated from the Fredholm determinant.

The purpose of the present work is to prove a conjecture made in Ref. [1] that identified the Fredholm determinant of a Dirac operator acting in one spatial dimension with a finite-dimensional determinant. The latter was constructed from the matrix that takes the solution of the Dirac equation at one point in space and transfers it to another position. The great virtue of the identification made in the conjecture is that the determinant of an infinite-dimensional matrix may be found by computations on matrices whose dimension is low; namely that of the matrices appearing in the Dirac equation. In the case where the potential appearing in the Hamiltonian has a piecewise constant variation in space (always valid to some level of approximation), the Fredholm determinant is very simply calculable in closed form by multiplying together matrices of low order.

This work is arranged as follows. In Section 2 the conjecture on Fredholm determinants is motivated and in Section 3 it is proved. Section 4 deals with the form of the Fredholm determinant when the length of the system tends to infinity and a related result is derived in an appendix. The paper is concluded by a short discussion in Section 5.

Throughout the work, we set the natural constants of c and \hbar to unity and freely use the relationship between the trace and determinant of a matrix: $\text{tr} \ln \equiv \ln \det$ for both finite and infinite matrices (first quantised operators).

² For some systems an expression for the free energy may be given directly in terms of the Fredholm determinant associated with the problem. For example, in Ref. [2], the free energy of a vortex in a type II superconductor is expressed in this way.

³ Throughout this work we use Det and Tr to denote the determinant and trace of infinite matrices (first quantised operators), whereas for 2×2 matrices we denote these quantities by \det and tr .

2. MOTIVATION OF THE CONJECTURE

For completeness we present the motivation of the conjecture made in Ref. [1]. Let us take, for the Hamiltonians H and H_0 , appearing in Eq. (1.1),

$$H = \sigma_1 p_x + \sigma_3 m + V(x) \quad (2.1a)$$

$$H_0 = \sigma_1 p_x + \sigma_3 m, \quad (2.1b)$$

where σ_i , $i=1, 2, 3$, are the usual Pauli matrices, two of which act as a 2×2 representation of the Dirac matrices in one spatial dimension. p_x is the momentum operator for motion along the x -axis, m is the mass of the particle, and the potential energy, $V(x)$, is assumed to have no matrix structure.

Let us consider how the eigenvalues of the operator H are determined. To make the problem well defined we must impose a boundary condition and we take the eigenfunctions to be periodic over an interval $2L$. Thus, in coordinate representation, the eigenvalues of H are the values of z such that the two component eigenfunction ψ obeys *both*⁴

$$[-i \partial_x \sigma_1 + m \sigma_3 + V(x)] \psi(x) = z \psi(x) \quad (2.2a)$$

and

$$\psi(L) = \psi(-L). \quad (2.2b)$$

Since Eq. (2.2a) is a linear differential equation, we can use this equation to relate $\psi(x)$ to $\psi(-L)$ and generally we find that there is a 2×2 matrix $M(x; z)$ that transports a solution from $-L$ to x . It obeys

$$[-i \partial_x \sigma_1 + m \sigma_3 + V(x) - z] M(x; z) = 0, \quad M(-L; z) = 1. \quad (2.3)$$

In terms of this matrix,

$$\psi(x) = M(x; z) \psi(-L). \quad (2.4)$$

The quantization condition, Eq. (2.2b) can, using Eq. (2.4), be cast in the form

$$[1 - M(L; z)] \psi(-L) = 0 \quad (2.5)$$

⁴ Compared with Ref. [1], we have, here, taken a different representation for the Dirac matrices and a different choice of space interval but these are differences in presentation only. Beyond this, the apparent difference of Eq. (2.1a) and its analogue in Ref. [1], namely Eq. (3.2b), is that this last equation had a spatially varying mass but no potential term. By contrast, Eq. (2.1a) has a constant mass but a non-zero potential. We can, however, eliminate the potential $V(x)$ from Eq. (2.1a) in favour of a spatially varying mass: With $\vartheta(x) = 2 \int^x dx' V(x')$ the unitary transformation $H \rightarrow \exp[i\vartheta(x) \sigma_1/2] H \exp[-i\vartheta(x) \sigma_1/2]$ generates a Hamiltonian with the mass term $m \sigma_3 \exp[-i\vartheta(x) \sigma_1]$ and no potential. For an appropriate choice of $V(x)$, the piecewise constant mass term equivalent to that of Ref. [1] can be obtained. Thus a consideration of the Hamiltonian H of Eq. (2.1a) with general potential $V(x)$ is sufficient to cover the special case considered in Ref. [1]. To cover a wider class of problems, it is possible to include in H , spatial variations of both the mass and the potential but we do not do so here.

and the condition for a non-vanishing eigenfunction is

$$\det[1 - M(L; z)] = 0. \tag{2.6}$$

The eigenvalues E_n are the values of z satisfying Eq. (2.6). The corresponding condition for the eigenvalues of H_0 , obtained by setting $V(x) = 0$, is written as

$$\det[1 - M_0(L; z)] = 0. \tag{2.7}$$

The conjecture made in Ref. [1] was that for general, complex, values of z , the quantity $d(z)$ defined by

$$d(z) = \frac{\det[1 - M(L; z)]}{\det[1 - M_0(L; z)]} \equiv \det\left(\frac{1 - M(L; z)}{1 - M_0(L; z)}\right) \tag{2.8}$$

equals the Fredholm determinant $D(z)$ of Eq. (1.1) when the same boundary condition is imposed on the eigenfunctions.

The arguments given in Ref. [1] were basically that (i) $d(z)$ has the same zeros and poles as $D(z)$ and (ii) as $|z| \rightarrow \infty$, $d(z)$ has the same behaviour as $D(z)$. In the following section we give a proof that $d(z) = D(z)$.

3. PROOF OF CONJECTURE

In the case of the Schrödinger operators, one possible method of obtaining the Fredholm determinant is to determine the Greens function $(z - H)^{-1}$ and then relate this to the Fredholm determinant [4]. We follow the same route and define

$$G = \frac{1}{z - H}, \quad G_0 = \frac{1}{z - H_0}, \quad z \text{ complex}; \tag{3.1}$$

then

$$\begin{aligned} \text{Tr}(G - G_0) &= \text{Tr}\left(\frac{1}{z - H} - \frac{1}{z - H_0}\right) = \frac{d}{dz} \text{Tr} \ln\left(\frac{z - H}{z - H_0}\right) \\ &= \frac{d}{dz} \ln \text{Det}\left(\frac{z - H}{z - H_0}\right) \equiv \frac{d}{dz} \ln D(z). \end{aligned} \tag{3.2}$$

We shall find G and apply the above equation.

(i) Determination of the Greens function

We regard operators such as H as acting in an abstract space of bras and kets; thus the coordinate representation of G is $\langle x | G | x' \rangle \equiv G(x; x')$ (the 2×2 matrix structure of G is left implicit). Since $(z - H)G = 1$ it follows that $G(x; x')$ obeys

$$(z - [-i\sigma_1 \partial_x + m\sigma_3 + V(x)]) G(x, x') = \delta(x - x'). \tag{3.3}$$

By multiplying this by $-i\sigma_1$ we obtain

$$[\partial_x - N(x)] G(x, x') = -i\sigma_1 \delta(x - x'), \quad (3.4a)$$

where

$$N(x) = i\sigma_1 [z - m\sigma_3 - V(x)]. \quad (3.4b)$$

$G_0(x, x')$ is obtained by setting $V(x) = 0$, in which case $N(x)$ is replaced by

$$N_0 = i\sigma_1 [z - m\sigma_3]. \quad (3.5)$$

The periodic boundary condition on the eigenfunctions, Eq. (2.2b), requires

$$G(L, x') = G(-L, x'). \quad (3.6)$$

The Greens function can be constructed using the 2×2 matrix $M(x; z)$ of Eq. (2.3) which, henceforth, we write as $M(x)$. In terms of $N(x)$, the matrix $M(x)$ obeys

$$[\partial_x - N(x)] M(x) = 0, \quad M(-L) = 1. \quad (3.7)$$

In terms of a 2×2 matrix, A , that is independent of x , we write

$$\begin{aligned} G(x, x') &= M(x) A & x < x' \\ &= M(x) M^{-1}(L) A & x > x' \end{aligned} \quad (3.8)$$

and this automatically obeys Eq. (3.6).

The delta function in Eq. (3.4a) leads to the discontinuity

$$G(x; x') \Big|_{x=x'_-}^{x=x'_+} = -i\sigma_1 \quad (3.9)$$

and combining this with Eq. (3.8) leads to a determination of the matrix A and hence the Greens function. We find

$$G(x; x') = -\frac{i}{2} M(x) \left(\frac{1 + M(L)}{1 - M(L)} + \text{sgn}(x - x') \right) M^{-1}(x') \sigma_1. \quad (3.10)$$

(ii) *Expressing the Fredholm Determinant in Terms of the Matrix $M(L)$*

Using Eqs. (3.2) and (3.10) leads to⁵

$$\begin{aligned} \frac{d}{dz} \ln D(z) &= -\frac{i}{2} \text{tr} \int_{-L}^L dx \left\{ M^{-1}(x) \sigma_1 M(x) \left(\frac{1 + M(L)}{1 - M(L)} \right) \right. \\ &\quad \left. - M_0^{-1}(x) \sigma_1 M_0(x) \left(\frac{1 + M_0(L)}{1 - M_0(L)} \right) \right\}. \end{aligned} \quad (3.11)$$

⁵ The trace, Tr, appearing in Eq. (3.2) is taken as $\text{Tr}(\dots) \equiv \text{tr} \int_{-L}^L dx \langle x | \dots | x \rangle$.

In Appendix A, it is shown that

$$i \int_{-L}^L dx M^{-1}(x) \sigma_1 M(x) = M^{-1}(L) \frac{dM(L)}{dz} \tag{3.12}$$

and using this leads to

$$\begin{aligned} \frac{d}{dz} \ln D(z) = & -\frac{1}{2} \operatorname{tr} \left\{ \frac{dM(L)}{dz} \left(\frac{1}{M(L)} + \frac{2}{1-M(L)} \right) \right. \\ & \left. - \frac{dM_0(L)}{dz} \left(\frac{1}{M_0(L)} + \frac{2}{1-M_0(L)} \right) \right\}. \end{aligned} \tag{3.13}$$

Taking the trace of Eq. (3.12) indicates that $\operatorname{tr}[M^{-1}(L) dM(L)/dz] = 0$ and we can write Eq. (3.13) as

$$\begin{aligned} \frac{d}{dz} \ln D(z) &= \frac{d}{dz} \operatorname{tr} \ln \left(\frac{1-M(L)}{1-M_0(L)} \right) \\ &= \frac{d}{dz} \ln \det \left(\frac{1-M(L)}{1-M_0(L)} \right) \\ &\equiv \frac{d}{dz} \ln d(z). \end{aligned} \tag{3.14}$$

Thus $\ln D(z)$ and $\ln d(z)$ differ, at most, by an additive constant. In Appendix B we show that in the limit of large $|z|$, the leading behaviour of $\ln D(z)$ is identical to that of $\ln d(z)$, allowing us to conclude that $D(z) = d(z)$; i.e.,

$$\operatorname{Det} \left(\frac{z-H}{z-H_0} \right) = \det \left(\frac{1-M(L)}{1-M_0(L)} \right), \tag{3.15}$$

and this is the result we sought to prove.

4. LARGE L LIMIT

In Ref. [1], a problem equivalent to a Dirac equation with a periodic potential was treated in some detail. To complement that calculation we shall consider here a potential that is different from zero only in a finite region of space and investigate the form of the Fredholm determinant $D(z) \equiv d(z)$ in the “infinite volume limit,” $L \rightarrow \infty$.

To begin, we define a 2×2 matrix $U(x)$ via

$$U(x) = e^{-xN_0} M(x) e^{-LN_0} \tag{4.1}$$

and using Eq. (3.7) it follows that $U(x)$ obeys

$$[\partial_x - W(x)] U(x) = 0, \quad U(-L) = 1, \quad (4.2a)$$

where

$$\begin{aligned} W(x) &= e^{-xN_0} [N(x) - N_0] e^{xN_0} \\ &= e^{-xN_0} [-i\sigma_1 V(x)] e^{xN_0}. \end{aligned} \quad (4.2b)$$

The solution to Eq. (4.2a) may be written as

$$U(x) = T_x \exp \left[\int_{-L}^x dx' W(x') \right] \quad (4.3)$$

where T_x denotes the “ x ordering operator” that orders the terms in ascending values of x ; terms with smaller values of x are placed to the right of terms with larger values. Equations (4.1) and (4.3) allow us to write

$$M(L) = e^{LN_0} U(L) e^{LN_0} \quad (4.4)$$

so

$$D(z) = \det \left(\frac{1 - e^{LN_0} U(L) e^{LN_0}}{1 - e^{2LN_0}} \right). \quad (4.5)$$

To determine the behaviour of $D(z)$ as $L \rightarrow \infty$, we note that e^{2LN_0} has two eigenvalues; one exponentially large,⁶ the other exponentially small, and the limiting form of $D(z)$ is determined by projection onto the space of the large eigenvalue. To isolate the contribution of the large eigenvalue we define⁷

$$\lambda = \sqrt{m^2 - z^2} \quad (4.6)$$

and note that

$$P_+ = \frac{1}{2}(1 + N_0/\lambda), \quad P_- = \frac{1}{2}(1 - N_0/\lambda), \quad (4.7)$$

are a complete and orthogonal set of projection operators that project onto the large and small eigenvalues of e^{2LN_0} :

$$P_+ + P_- = 1, \quad P_+ P_- = 0 \quad (4.8a)$$

$$e^{2LN_0} P_{\pm} = e^{\pm 2L\lambda} P_{\pm}. \quad (4.8b)$$

⁶ For $|z| > m$, the limit $L \rightarrow \infty$ does not exist unless $\text{Im}(z) \neq 0$, signalling the appearance of cuts on the real z axis along $(-\infty, -m)$ and (m, ∞) . These indicate the presence of a continuum of scattering states at $(\text{energies})^2 \geq m^2$. The imaginary part of z ensures that e^{2LN_0} has an eigenvalue that grows exponentially with L . For $|z| < m$, e^{2LN_0} always has an eigenvalue that grows exponentially with L .

⁷ The square root is defined on the complex z plane cut along the negative real axis; the branch selected has $\text{Re}(\lambda) > 0$. Thus for large $|z|$, $\lambda \approx -i \text{sgn}(\text{Im}(z)) z + \dots$

It is proved in Appendix C that the determinant of any 2×2 matrix A may be written as

$$\det A = \text{tr}(P_+ A) \text{tr}(P_- A) - \text{tr}(P_+ A P_- A) \tag{4.9}$$

and applying this to Eq. (4.5) gives, for large L ,

$$D(z) \approx \frac{-e^{2L\lambda} \text{tr}[P_+ U(L)] + O(e^0)}{-e^{2L\lambda} + O(e^0)}. \tag{4.10}$$

Thus

$$\lim_{L \rightarrow \infty} D(z) \equiv D_\infty(z) = \text{Tr} \left[P_+ \left(T_x \exp \left[\int_{-\infty}^{\infty} dx W(x) \right] \right) \right], \tag{4.11}$$

a result that may be useful when perturbation theory is applicable.

We note that for $|z| \rightarrow \infty$, along a ray through the origin, the right-hand side of Eq. (4.11) collapses to $\exp[i \text{sgn}(\text{Im}(z)) \int_{-\infty}^{\infty} dx V(x)]$, a result compatible with Eq. (B.7).

In Appendix D we sketch how $D_\infty(z)$ is related to an analytic continuation of the transmission amplitude in the scattering problem.

5. DISCUSSION

In this work we have proved Eq. (3.15) which gives the relation between a Fredholm determinant for a Dirac operator and a finite determinant constructed from the matrix $M(x)$ that transfers a solution of the Dirac equation from one point in space to another.

By a straightforward use of the methods presented here we could extend the results to Dirac equations with matrices of order greater than two and allow more general spatial variations by upgrading the potential $V(x)$ to a matrix. There may also be advantages in extending the results of this work to boundary conditions other than periodic; however, we have not pursued this matter.

Let us end this work by illustrating the simplification that follows from the main result of this work, Eq. (3.15), by sketching the calculation required to determine the Fredholm determinant for a piecewise constant potential. We consider the square potential well

$$\begin{aligned} V(x) &= -V_1, & |x| < a \\ &= 0, & |x| > a. \end{aligned} \tag{5.1}$$

The matrix $M(L)$ following from Eq. (3.6) takes the form

$$\exp[(L-a)N_0] \exp[2aN_1] \exp[(L-a)N_0], \tag{5.2a}$$

where

$$N_0 = i\sigma_1 [z - m\sigma_3], \quad N_1 = i\sigma_1 [z - m\sigma_3 + V_1]. \quad (5.2b)$$

Then, since $\det(\exp[(L-a)N_0]) = 1$, we have, using Eq. (3.15),

$$\text{Det} \left(\frac{z-H}{z-H_0} \right) = \det \left(\frac{1 - \exp[2(L-a)N_0] \exp[2aN_1]}{1 - \exp[2LN_0]} \right). \quad (5.3)$$

The evaluation of the determinant on the right-hand side of this equation is straightforward and yields an expression in closed form for the Fredholm determinant.

APPENDIX A: PROOF OF THE IDENTITY

$$i \int_{-L}^L dx M^{-1}(x) \sigma_1 M(x) = M^{-1}(L) (dM(L)/dz)$$

In this appendix, we prove the above identity which is used in Section 3 of this work. We begin with Eq. (3.7) which we differentiate with respect to z :

$$[\partial_x - N(x)] \frac{dM(x)}{dz} - \frac{dN(x)}{dz} M(x) = 0. \quad (A.1)$$

The explicit z dependence of $N(x)$, given in Eq. (3.4b), results in

$$[\partial_x - N(x)] \frac{dM(x)}{dz} = i\sigma_1 M(x). \quad (A.2)$$

We write this equation as

$$[\partial_x - N(x)] M(x) M^{-1}(x) \frac{dM(x)}{dz} = i\sigma_1 M(x), \quad (A.3)$$

and using Eq. (3.7) yields

$$M(x) \partial_x \left(M^{-1}(x) \frac{dM(x)}{dz} \right) = i\sigma_1 M(x) \quad (A.4)$$

or

$$\partial_x \left(M^{-1}(x) \frac{dM(x)}{dz} \right) = iM^{-1}(x) \sigma_1 M(x). \quad (A.5)$$

Integrating this equation from $-L$ to L yields

$$M^{-1}(L) \frac{dM(L)}{dz} - M^{-1}(-L) \frac{dM(-L)}{dz} = i \int_{-L}^L dx M^{-1}(x) \sigma_1 M(x). \quad (A.6)$$

The second term on the left side of this equation vanishes since $M(-L)$ is the unit matrix⁸ and we obtain the desired result.

APPENDIX B: LARGE $|z|$ BEHAVIOUR OF $\ln D(z)$ AND $\ln d(z)$

We have $D(z) \equiv \text{Det}((z - H)/(z - H_0))$; $d(z) \equiv \det((1 - M(L))/(1 - M_0(L)))$ and in Eq. (3.14) we established that $\ln D(z)$ and $\ln d(z)$ differ, at most by an additive constant. By examining the large $|z|$ behaviour of each of these and comparing the results, we can pin down the value of the additive constant.

It should be noted that both H and H_0 have (infinitely many) eigenvalues along the real axis. As a consequence, $D(z)$ will have an infinite sequence of zeros and poles along this line and, in general, a limit of $D(z)$ only exists when z becomes infinitely far from the real axis. Thus the by limit $|z| \rightarrow \infty$, we mean that z tends to infinite along a ray through the origin that does not coincide with the real axis.⁹

(i) Behaviour of $\ln D(z)$

We have

$$H = H_0 + V \tag{B.1}$$

so

$$\begin{aligned} \ln D(z) &= \ln \text{Det} \left(\frac{z - H}{z - H_0} \right) = \text{Tr} \ln \left(\frac{z - H}{z - H_0} \right) \\ &= \text{Tr} \ln(1 - G_0 V) = -\text{Tr}[G_0 V] - \dots, \end{aligned} \tag{B.2}$$

the higher order terms vanishing as $|z| \rightarrow \infty$.

Then

$$\text{Tr}[G_0 V] = \int_{-L}^L dx \text{tr}[G_0(x, x)] V(x) \tag{B.3}$$

and from Eq. (3.10) we have

$$\text{tr}[G_0(x, x)] = -\frac{i}{2} \text{tr} \left\{ M_0^{-1}(x) \sigma_1 M_0(x) \left(\frac{1 + M_0(L)}{1 - M_0(L)} \right) \right\}. \tag{B.4}$$

$M_0(x)$ follows from Eqs. (3.4a) and (3.5):

$$M_0(x) = \exp[(x + L)N_0], \tag{B.5}$$

⁸ Strictly, we should write the second term on the left side of Eq. (A.6) as the limit $\text{Lim}_{x \rightarrow -L} M^{-1}(x) dM(x)/dz$. For $x \sim -L$, however, $M(x) \approx 1 + \int_{-L}^x dx' N(x')$ and, using Eq. (3.4b) for $N(x)$, we find $dM(x)/dz \approx i\sigma_1(x + L)$, indicating that the naive assignment $dM(-L)/dz = 0$ is correct.

⁹ Where necessary, the limit along a ray is taken by writing $z = \mu e^{i\vartheta}$, where μ and ϑ are real, $\vartheta \neq 0, \pi$, and allowing μ to tend to $+\infty$.

and it may be verified that the $|z| \rightarrow \infty$ limit of Eq. (B.4) is equivalent to setting the mass of the particle, m , to zero (cf. Ref. [3]). A straightforward calculation shows that

$$\lim_{\substack{|z| \rightarrow \infty \\ \text{along ray}}} \operatorname{tr}[G_0(x; x)] = -i \operatorname{sgn}(\operatorname{Im}(z)). \quad (\text{B.6})$$

Thus

$$\lim_{\substack{|z| \rightarrow \infty \\ \text{along ray}}} \ln D(z) = i \operatorname{sgn}(\operatorname{Im}(z)) \int_{-L}^L dx V(x). \quad (\text{B.7})$$

(ii) *Behaviour of $\ln d(z)$*

To determine the behaviour of $\ln d(z)$ for large $|z|$, it is allowable to set $m = 0$ in $d(z)$ and we do this first. When this is done the matrices $M(L)$ and $M_0(L)$ depend only on the unit matrix and σ_1 and the determinant in $d(z)$ is trivially evaluated. Last, the limit $|z| \rightarrow \infty$ can be taken with the result:

$$\lim_{\substack{|z| \rightarrow \infty \\ \text{along ray}}} \ln d(z) = i \operatorname{sgn}(\operatorname{Im}(z)) \int_{-L}^L dx V(x). \quad (\text{B.8})$$

Thus in the limit of large $|z|$, $\ln D(z)$ and $\ln d(z)$ coincide.

APPENDIX C: PROOF OF $\det A = \operatorname{tr}(P_+ A) \operatorname{tr}(P_- A) - \operatorname{tr}(P_+ A P_- A)$

The above relation holds for an arbitrary 2×2 matrix A , where P_{\pm} are a complete and orthogonal set of projection operators (i.e., $P_+ + P_- = 1$; $P_+ P_- = 0$). To prove it we use the following result, valid for an arbitrary 2×2 matrix:

$$\det A = \frac{1}{2} [(\operatorname{tr} A)^2 - \operatorname{tr}(A^2)]. \quad (\text{C.1})$$

Applying Eq. (C.1) to $\det[(P_+ + P_-) A]$ yields

$$\begin{aligned} \det[(P_+ + P_-) A] &= \operatorname{tr}(P_+ A) \operatorname{tr}(P_- A) - \operatorname{tr}(P_+ A P_- A) \\ &\quad + \frac{1}{2} [(\operatorname{tr}[P_+ A])^2 - \operatorname{tr}[(P_+ A)^2]] \\ &\quad + \frac{1}{2} [(\operatorname{tr}[P_- A])^2 - \operatorname{tr}[(P_- A)^2]]. \end{aligned} \quad (\text{C.2})$$

Use of Eq. (C.1) enables us to write this as

$$\begin{aligned} \det[(P_+ + P_-) A] &= \operatorname{tr}(P_+ A) \operatorname{tr}(P_- A) - \operatorname{tr}(P_+ A P_- A) \\ &\quad + \det[P_+ A] + \det[P_- A]. \end{aligned} \quad (\text{C.3})$$

Then if $P_+ + P_- = 1$ and $P_+ P_- = 0$ it can be proved that any non-zero 2×2 matrices that satisfy these have vanishing determinant, i.e., $\det[P_\pm] = 0$ (this corresponds to each projection operator annihilating a vector, and therefore having an eigenvalue of zero). Consequently the last two terms on the right of Eq. (C.3) vanish and we obtain the required result:

$$\det A = \text{tr}(P_+ A) \text{tr}(P_- A) - \text{tr}(P_+ A P_- A). \tag{C.4}$$

There are versions of Eq. (C.1) applicable to square matrices of arbitrary order and corresponding generalizations of Eq. (C.4).

Note that if the projection operators are written as

$$P_\pm = \frac{1}{2}(1 \pm \boldsymbol{\alpha} \cdot \boldsymbol{\sigma}) \quad \text{with} \quad \boldsymbol{\alpha} \cdot \boldsymbol{\alpha} = 1 \tag{C.5}$$

then for the vector $\boldsymbol{\alpha}$ real, Eq. (C.4) corresponds to the usual result for the determinant of A , evaluated in the orthogonal basis formed from right and left eigenvectors of $\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}$ which are Hermitian adjoints of one another. For $\boldsymbol{\alpha}$ complex (the case applicable to the present work), the right and left eigenvectors of $\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}$ are no longer Hermitian adjoints of one another and Eq. (C.4) may be viewed as a generalisation of the usual formula for a determinant to more general bases.

APPENDIX D: RELATION OF $D_\infty(z)$ TO AN ANALYTIC CONTINUATION OF THE TRANSMISSION AMPLITUDE

In Ref. [4], the Fredholm determinant for a Schrödinger operator was related to an analytic continuation of the transmission amplitude in a scattering problem. The eigenfunctions in the problem were, implicitly, assumed to be bounded on the infinite line $(-\infty, \infty)$. We have reproduced the analysis for the case of the Dirac operator studied in this work. The similarities with the Schrödinger case are sufficiently great that we shall just sketch the details.

We begin with Eq. (3.1), applied to the case of a potential that is non-zero in only a finite region of the infinite line. With λ as defined in Eq. (4.6) we let $f_\pm(x)$ be independent spinor eigenfunctions obeying Eq. (2.2a) with the asymptotic form

$$\lim_{x \rightarrow \pm\infty} f_\pm(x; z) \approx t(z) V_\pm e^{\mp \lambda x} \tag{D.1a}$$

$$\lim_{x \rightarrow \mp\infty} f_\pm(x; z) \approx V_\pm e^{\mp \lambda x} + r_\pm(z) V_\mp e^{\pm \lambda x}, \tag{D.1b}$$

where

$$V_\pm = \begin{pmatrix} z + m \\ \pm i\lambda \end{pmatrix}, \tag{D.2}$$

$t(z)$ and $r_{\pm}(z)$ are complex numbers corresponding to the analytic continuation of transmission and reflection amplitudes into the complex energy plane. It can be proved that $t(z)$ is the same for both f_+ and f_- .

With T denoting the transpose of a matrix/spinor, we define for two spinors f_1 and f_2 , the quantity

$$W[f_1, f_2] = -if_1^T \sigma_2 f_2 \quad (\text{D.3})$$

which is analogous to the Wronskian in Schrödinger theory. The Greens function $G(x; x')$ that remains bounded as $|x| \rightarrow \infty$ may be shown to be

$$\begin{aligned} G(x; x') &= \frac{if_+^T(x; z) f_-^T(x'; z) \sigma_3}{W[f_-(x; z), f_+(x; z)]}, & x > x' \\ &= \frac{if_-^T(x; z) f_+^T(x'; z) \sigma_3}{W[f_-(x; z), f_+(x; z)]}, & x < x' \end{aligned} \quad (\text{D.4})$$

and leads to

$$\text{tr } G(x; x) = \frac{if_-^T(x; z) \sigma_3 f_+(x; z)}{W[f_-(x; z), f_+(x; z)]} \quad (\text{D.5})$$

($\text{tr } G(x; x')$ is continuous at $x' = x$).

It may be straightforwardly proven that

$$\partial_x W[f_-(x; z'), f_+(x; z)] = (z' - z) if_-^T(x; z') \sigma_3 f_+(x; z); \quad (\text{D.6})$$

thus

$$\text{tr } G(x; x) = \frac{\partial_x W[df_-(x; z)/dz, f_+(x; z)]}{W[f_-(x; z), f_+(x; z)]}. \quad (\text{D.7})$$

We now employ Eq. (3.2) which yields

$$\begin{aligned} \frac{d}{dz} \ln D_{\infty}(z) &= \text{Tr}(G - G_0) \\ &= \lim_{L \rightarrow \infty} \int_{-L}^L dx \text{tr}[G(x; x) - G_0(x; x)] \end{aligned} \quad (\text{D.8})$$

and using the preceding equations and their analogues for $V(x) = 0$ results in

$$\frac{d}{dz} \ln D_{\infty}(z) = \frac{d}{dz} \ln[t^{-1}(z)]. \quad (\text{D.9})$$

From considerations of this equation for $|z| \rightarrow \infty$ (cf. Appendix B), it may be verified that

$$D_{\infty}(z) = t^{-1}(z). \quad (\text{D.10})$$

This is the relation between the Fredholm determinant and the (analytic continuation of) the transmission amplitude.

This work is dedicated to the memory of my father.

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Note added in proof. The calculations presented here for Dirac operators have been extended to Bogoliubov operators. Details of this work are given in D. WAXMAN, *Phys. Rev. Lett.* **72** (1994), 570.

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