J. Phys. C: Solid State Phys. 19 (1986) L547–L550. Printed in Great Britain

LETTER TO THE EDITOR

Anomalous currents and fractional charge associated with domain walls in superfluid ^{3}He-A films

D Pattarini and D Waxman
Physics Division, School of Mathematical and Physical Sciences, The University of Sussex, Brighton BN1 9QH, UK

Received 23 June 1986

Abstract. Calculations are presented for the (zero-temperature) mass current in a thin film of superfluid ^{3}He-A containing a polar domain wall. While there is no obvious analogue of twist ($\hat{l}\cdot\nabla\times\hat{l}$) in the film, a term in the current is found that is analogous to the twist term in bulk A phase. The term found possesses a δ-function singularity; however, the total current is non-singular. It is additionally shown that the total current may be expressed in the language of fractional fermionic charge.

Topological solitons (domain walls) in superfluid ^{3}He-A have recently been shown to cause modifications in the quasi-particle excitation spectrum and result in the existence of bound quasi-particle states (on the domain wall) (Ho *et al* 1984). Connected with these effects are the manifestation of certain field theory ideas such as anomalies (Volovik 1985) and fractional fermion number (Ho *et al* 1984, Stone *et al* 1985). Mathematically, the connection with field theories arises since the Gorkov–Nambu equations for the Green functions are, on linearisation, closely related to the Dirac equation.

It is the object of this Letter to extend the investigation of Nakahara (1986) to the existence of the above phenomena in a system that is not simply bulk ^{3}He–A. The system consists of a thin film of ^{3}He in which a polar-phase domain wall interpolates between two regions in the A phase (Ohmi *et al* 1982). The work of Nakahara (1986) indicates that the domain wall does modify the excitation spectrum in a similar way to what was found by Ho *et al* (1984) for domain walls in bulk ^{3}He-A. The modification includes that of the spectrum becoming asymmetric as a function of quasi-particle momentum. This is perhaps surprising since in the texture of Ohmi *et al* (1982) there is no obvious analogue of the 'twist' $\hat{l}\cdot\nabla\times\hat{l}$ associated with the domain walls in bulk ^{3}He-A—which, there, is found to be responsible for the spectral asymmetry.

Here we concern ourselves solely with the mass current as viewed from the two existing viewpoints:

(i) of having an anomalous term (Volovik 1985) associated with a normal fluid component (which results from the gap vanishing along certain quasi-particle directions);
(ii) of having its origin as a sum of fractional fermionic charge densities associated with different quasi-particle momenta on the Fermi surface (Stone *et al* 1985).

We recall that in ^{3}He-A at $T = 0$ the mass current is (Mermin and Muzikar 1980)

$$\boldsymbol{g} = \boldsymbol{g}_{\mathrm{A}} + \boldsymbol{g}_{\mathrm{B}} + \boldsymbol{g}_{\mathrm{C}} \tag{1}$$

$$\boldsymbol{g}_{\mathrm{A}} = \rho \boldsymbol{v}_{\mathrm{s}} \tag{1a}$$

$$\boldsymbol{g}_{\mathrm{B}} = (1/4M)\boldsymbol{\nabla} \times (\rho \hat{\boldsymbol{l}}) \tag{1b}$$

$$\boldsymbol{g}_{\mathrm{C}} = -(1/2M)\rho \hat{\boldsymbol{l}}(\hat{\boldsymbol{l}} \cdot \boldsymbol{\nabla} \times \hat{\boldsymbol{l}}). \tag{1c}$$

It was (1c) that was identified by Volovik (1985) as being anomalous. There is an analogue of (1c) in the film with a domain wall proposed by Ohmi *et al* (1982). This film is described by

$$\boldsymbol{\varphi}_1 = \hat{\boldsymbol{e}}_x \qquad \text{(unit vector in the } x \text{ direction)} \tag{2}$$

$$\boldsymbol{\varphi}_2 = \tanh(x/r)\hat{\boldsymbol{e}}_y \equiv t\hat{\boldsymbol{e}}_y \tag{3}$$

with the order parameter

$$C = C_1 + \mathrm{i}\, C_2 = (\tfrac{1}{2})^{1/2}(\boldsymbol{\varphi}_1 + \mathrm{i}\boldsymbol{\varphi}_2) \cdot \boldsymbol{p}. \tag{4}$$

(We follow the notation of Mermin and Muzikar (1980), and orientate our coordinates in a similar way in Nakahara (1986).)

Equations (2) and (3) describe a thin film lying in the z plane which corresponds to the A phase for $|x| \to \infty$ with $\hat{\boldsymbol{l}} \| \hat{\boldsymbol{e}}_z$ for $x \to +\infty$, $\hat{\boldsymbol{l}} \| -\hat{\boldsymbol{e}}_z$ for $x \to -\infty$ and a polar domain wall at $x = 0$.

The results (1a), (1b), (1c) are only applicable when $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$ are orthogonal unit vectors (i.e. in the A phase). The presence of $\tanh(x/r)$ in (3) results in $\boldsymbol{\varphi}_2$ not being a unit vector in the vicinity of $x = 0$. Consequently the phase deviates away from the A phase (into the polar phase) and (1a), (1b), (1c) do not apply. To find the current in this case (at $T = 0$) it is necessary to return to the more general expressions given in Mermin and Muzikar (1980); their equation (2.27) for the current yields ($\hbar = 1$)

$$\boldsymbol{g}_{\mathrm{A}} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \rho^{(0)} \operatorname{Im}\left(\frac{1}{C}\boldsymbol{\nabla}_r C\right)\Theta(|C|^2 - \lambda) \tag{5}$$

$$\boldsymbol{g}_{\mathrm{B}} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \boldsymbol{p}\boldsymbol{\nabla}_r \cdot \left\{\rho^{(0)} \operatorname{Im}\left(\frac{1}{C}\boldsymbol{\nabla}_p C\right) \Theta(|C|^2 - \lambda)\right. \tag{6}$$

$$\boldsymbol{g}_{\mathrm{C}} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \boldsymbol{p}\rho^{(0)} \operatorname{Im}[C, C^*]\delta(|C|^2 - \lambda) \tag{7}$$

in which the limit $\lambda \to 0_+$ is implicit and [,] denotes a Poisson bracket.

On substituting (4) into (7) we find after some straightforward algebra ($t = \tanh(x/r)$, $t' \equiv \mathrm{d}t/\mathrm{d}x$)

$$\boldsymbol{g}_{\mathrm{C}} = \lim_{\lambda \to 0_+} 2t'\hat{\boldsymbol{e}}_y \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \Theta(p_{\mathrm{F}}^2 - p^2)\delta(p_x^2 + (tp_y)^2 - \lambda). \tag{8}$$

Making the substitutions

$$z_i = p_i/p_{\mathrm{F}} \qquad i = x, y, z \tag{9}$$

$$\gamma^2 = \lambda/p_{\mathrm{F}}^2 \tag{10}$$

and integrating over z_1, z_3 gives

$$g_C = p_F^3 t' \hat{e}_y J \tag{11}$$

with

$$J = \lim_{\gamma \to 0_+} \int_0^L \frac{dz}{\pi^3} \frac{[(1-\gamma^2) - (1-t^2)z^2]^{1/2} z^2}{[\gamma^2 - (tz)^2]^{1/2}} \tag{12a}$$

and

$$L = \mathrm{Min}\left\{\frac{\gamma}{|t|}, \left(\frac{1-\gamma^2}{1-t^2}\right)^{1/2}\right\}. \tag{12b}$$

The integral for J requires careful treatment since the limits $\gamma \to 0_+$, $t \to 0$ do not commute, suggesting a singular behaviour at $t = 0$ (i.e. $x = 0$). Our assertion is that $J(t) \propto \delta(t)$. This can be verified by considering

$$\int_{-R}^{R} J(t)\, dt.$$

For R large compared with γ we find after some algebra that the result is finite and essentially independent of R and γ. The exact result for J is thus found to be

$$J(t) = (1/3\pi^2)\delta(t). \tag{13}$$

On combining (11) and (13), and denoting the fluid mass density by

$$\rho = M p_F^3 / 3\pi^2 \tag{14}$$

we obtain for g_C

$$g_C = (\rho/M) t' \delta(t) \hat{e}_y \equiv (\rho/M)\delta(x)\hat{e}_y. \tag{15}$$

This result is quite remarkable since it is singular†, independent of the domain wall width r, and is the outcome of a gradient expansion (which underlies (5), (6) and (7)).

If the overall mass current were singular we would not trust the result; however, when we go on to evaluate g_B we find (in the approximation where $\rho^{(0)}$ in (6) is replaced by $\Theta(p_F^2 - p^2)$)

$$g_B = -\frac{\rho}{2M}\frac{\partial}{\partial x}\left(\frac{\mathrm{sgn}(x)}{1 + \tanh(|x|/r)}\right)\hat{e}_y \tag{16}$$

that is

$$g_B = -\frac{\rho}{M}\delta(x)\hat{e}_y + \frac{\rho}{2Mr} e^{-2|x|/r}\hat{e}_y \tag{17}$$

and hence the δ-functions exactly cancel between g_B and g_C (g_A vanishes identically). In the interpretation of Volovik, g_B arises from a superfluid component and g_C from the normal component.

Note that the total mass current

$$g = \frac{\rho}{2Mr} e^{-2|x|/r}\hat{e}_y \tag{18}$$

is non-singular—as one would expect.

† A more elaborate calculation shows that the delta singularity gets smoothed out over length scales $\sim (r/p_F)^{1/2}$.

Let us now consider the question that naturally arises from the second viewpoint noted above. Can the current in the film be viewed as arising from a weighted sum over fractional fermionic densities? The answer to this question is yes, as may be seen by repeating the analysis of Stone *et al* (1985). The fermionic charge density associated with quasi-particles at the Fermi surface moving in direction $\hat{\boldsymbol{p}}$ is

$$n(x,\hat{\boldsymbol{p}}) = \operatorname{sgn}(p_x)\partial_x\theta(x,\hat{\boldsymbol{p}})/2\pi \tag{19}$$

where θ is given by

$$\tan\theta(x,\hat{\boldsymbol{p}}) = \hat{\boldsymbol{p}}\cdot\boldsymbol{\varphi}_2/\hat{\boldsymbol{p}}\cdot\boldsymbol{\varphi}_1 \tag{20}$$

(hats denote unit vectors).

The current is (cf (13) of Stone *et al* 1985†).

$$\boldsymbol{g} = \frac{p_{\mathrm{F}}^3}{2\pi^2}\int\frac{\mathrm{d}^2\hat{\boldsymbol{p}}}{4\pi}\,\hat{\boldsymbol{p}}(\hat{\boldsymbol{p}})_x\partial_x\theta(x,\hat{\boldsymbol{p}}) \tag{21}$$

(the integral is over the surface of the Fermi sphere).

A straightforward substitution of (2) and (3) into (21) gives the result (18) for the current.

To summarise, we have considered the mass current associated with a polar domain wall in a thin film of superfluid ^{3}He-A at $T = 0$. Despite there being no obvious 'twist' (chirality) in the system, an anomalous term in the current is found that is analogous to the twist term in bulk ^{3}He-A (albeit singular). An exact cancellation of this singularity with a term arising from the superfluid term in the current—leaving a non-singular total current—suggests that in this case the separation of the currents into g_{B} and g_{C} is unnatural. Finally, we have shown that the 'fractional fermionic' interpretation of currents holds beyond the pure A phase to 'mixed' phases of the type considered here.

Helpful discussions with David Bailin and Mikio Nakahara are acknowledged with thanks. One of us (DP) is in receipt of an SERC studentship.

References

Ho T L, Fulco J R, Schrieffer J R and Wilczek F 1984 *Phys. Rev. Lett.* **17** 1524
Mermin N D and Muzikar P 1980 *Phys. Rev.* B **21** 980
Nakahara M 1986 *J. Phys. C: Solid State Phys.* **19** L195
Ohmi T, Nakahara M, Tsuneto T and Fujita T 1982 *Prog. Theor. Phys.* **68** 1433
Stone M, Garg A and Muzikar P 1985 *Phys. Rev. Lett.* **21** 2328
Volovik G E 1985 *Pis'ma Zh. Eksp. Teor. Phys.* **42** 294 (Engl. Transl. 1985 *JETP Lett.* **42** 363)

† We have included a factor of two for spin.